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Spin cohomology

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Abstract

We explore differential and algebraic operations on the exterior product of spinor representations and their twists that give rise to cohomology, the spin cohomology. A linear differential operator d is introduced which is associated to a connection ∇ and a parallel spinor ζ, $\nabla\zeta = 0$, and the algebraic operators $D_{(p)}$ are constructed from skew-products of p gamma matrices. We exhibit a large number of spin cohomology operators and we investigate the spin cohomologies associated with connections whose holonomy is a subgroup of $SU(m)$, G_2, $Spin(7)$ and $Sp(2)$. In the $SU(m)$ case, we find that the spin cohomology of complex spin and $spin_c$ manifolds is related to a twisted Dolbeault cohomology. On Calabi-Yau type of manifolds of dimension $8k + 6$, a spin cohomology can be defined on a twisted complex with operator $d + D$ which is the sum of a differential and algebraic one. We compute this cohomology on six-dimensional Calabi-Yau manifolds using a spectral sequence. In the G_2 and $Spin(7)$ cases, the spin cohomology is related to the de Rham cohomology.

Keywords: Spin cohomology; Holonomy; Spinors

1. Introduction

On spin manifolds apart from the exterior derivative d and the associated de Rham complex $(\Lambda^*(M), d)$, one can define the Dirac operator $(\Delta(M), D)$, where $\Delta(M)$ is the spin bundle.[1] On complex manifolds, the Dirac operator decomposes as $D = \mathcal{D} + \bar{\mathcal{D}}$ and the spin representation can be graded such that $(\Delta(M), \bar{\mathcal{D}})$ can turn into a (graded) complex. The associated cohomology is called *spinor* cohomology [1].

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[1] We adopt the notation to denote a representation and its associated bundle with the same symbol, e.g. $\Delta = \Delta(\mathbb{R}^n)$ denotes the spin representation of $Spin(n)$ and $\Delta = \Delta(M)$ denotes also the spin bundle over M. In addition, we shall denote the bundles and their sections with the same symbol. Λ^* denotes the space of forms.

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In even dimensions, the complex Dirac spin represenation is reducible and decomposes as $\Delta = \Delta^- \oplus \Delta^+$. Apart from the spin representation Δ both the exterior power, $\mathcal{C} = \Lambda^*(\Delta^*)$ ($\mathcal{C}_\pm = \Lambda^*(\Delta_\pm)$), and the symmetric product, $\mathrm{Sym}^*(\Delta^*)$ ($\mathrm{Sym}^*(\Delta_\pm)$), of the dual spinor representation Δ^* ($\Delta_\pm = (\Delta^\pm)^*$) have found applications in various problems in physics. The former has applications in supermanifolds. In particular, all real supermanifolds that appear in the context of supersymmetry are isomorphic to $\mathcal{C}$ ($\mathcal{C}_\pm$) [2]. The latter is a model for the odd forms on supermanifolds and has appeared in the context of string theory [8] and the theory of deformations of the field equations of supersymmetric gauge theories and supergravity in superspace [6,7]. It turns out that the theory of deforming the field equations of the supersymmetric gauge theories and supergravity can turn into a cohomological problem on $\Lambda^* \otimes \mathrm{Sym}^*$ for the so called *spinorial* cohomology.

Motivated by these developments in physics, the aim of this paper is to investigate various cohomology operators that can be defined on $\mathcal{C}$, $\mathcal{C}_\pm$ and its various twistings. Let (M, g) be a spin manifold equipped with a spin connection ∇, which is *not* necessarily the Levi-Civita connection of the metric g. One can define a linear differential (spin) operator on $\mathcal{C}(M)$ or $\mathcal{C}_\pm(M)$ as

$$\mathrm{d}\phi = \zeta \Gamma^\mu \bar{\wedge} \nabla_\mu \phi, \quad \phi \in \mathcal{C}(M), \tag{1.1}$$

where $\bar{\wedge}$ is the wedge product in $\mathcal{C}$, ζ a cospinor and $\{\Gamma^\mu : \mu = 1, \ldots, \dim M\}$ are the gamma matrices. In many applications, ζ is taken to be a parallel cospinor with respect to ∇, $\nabla\zeta = 0$. As we shall see there are various cohomology theories that can be defined depending on the choice of spinor ζ, the connection ∇. The operator d can always be defined on $\mathcal{C}$. However, the restriction of d on $\mathcal{C}_\pm$ depends on the choice of ζ and the properties of the spinor inner products which in turn depend on the dimension of the manifold M. One of our aims is to investigate the conditions for d to be nilpotent, $d^2 = 0$. These conditions can be expressed in terms of restrictions on the cospinor ζ and on the curvature R of the connection ∇, in addition to $\nabla\zeta = 0$.

In addition to differential operators, we shall present a large number of algebraic cohomology operators $D_{(p)}$ on some twisted complexes like for example $\Lambda^*(M) \otimes \mathcal{C}(M)$ and $\Lambda^*(M) \otimes \mathcal{C}_\pm(M)$. Some of these are constructed from skew-products of p gamma matrices. We investigate the conditions for $D^2_{(p)} = 0$ and relate them into the symmetry properties of gamma matrices. The latter again depend on the dimension of the manifold M. In addition, we shall show that $D_{(p)}d + dD_{(p)} = 0$ and so the cohomology of $(\Lambda^* \otimes \mathcal{C}, d + D_{(p)})$ and $(\Lambda^* \otimes \mathcal{C}_\pm, d + D_{(p)})$ can be computed using a spectral sequence. We shall refer collectively to all of these cohomology theories with operators d, $D_{(p)}$ and $d + D_{(p)}$ as *spin* cohomologies.

We shall develop the general theory of spin cohomology. In particular, we shall compute the conditions on the curvature of the underlying manifold for $d^2 = 0$. We shall also explain the relation to parallel spinors.

Next, we shall focus on a certain class of parallel spinors. In particular, we shall consider manifolds which admit a spin connection ∇ induced from the tangent bundle with holonomy contained in the groups $SU(m)$ ($n = 2m$), $Sp(k)$ ($n = 4k$), $Spin(7)$ ($n = 8$) and G_2 ($n = 7$), where in parenthesis is the dimension of the manifold. A special class of examples of manifolds with spin cohomology are those that appear in the Berger list and admit parallel spinors [3], for non-simply connected manifolds see [4]. In these cases, ∇ is the Levi-Civita connection.

We shall show that for spin complex manifolds which admit a holomorphic connection ∇ with $\mathrm{hol}\nabla \subseteq SU(m)$, there are two differential spin cohomologies with operators d_1 and d_2 related to two parallel spinors of the connection ∇. We refer to these spin cohomologies as complex spin cohomologies. We shall show that d_1 and d_2 restrict on $\mathcal{C}_\pm$. In particular, one can construct complexes $(\mathcal{C}_+, d_1)$ and $(\mathcal{C}_+, d_2)$ for $\dim M = 8k + 2, 8k + 6$ and complexes $(\mathcal{C}_-, d_1)$ and

$(\mathcal{C}_-, d_2)$ for dim $M = 8k, 8k+4$. We give the Laplace operators associated with d_1 and d_2 using a $Spin(n)$-invariant inner product. We show that the complex spin cohomology of $(\mathcal{C}_-, d_2)$ in all dimensions is related to twisted Dolbeault cohomology. We extend this relation between this spin cohomology and Dolbeault cohomology to complex $spin_c$ manifolds as well. The complex spin cohomologies can be twisted with any holomorphic vector bundle. Apart form the differential complex spin cohomologies, there is an algebraic spin cohomology operator $D = D_{(1)}$ and the complex $\Lambda^{*,0} \otimes \mathcal{C}_-$ on all such manifolds of dimension $n = 8k + 6$ and $d_2 D + D d_2 = 0$. The cohomology of $(\Lambda^{*,0} \otimes \mathcal{C}_-, d_2 + D)$ can be computed using a spectral sequence. As an example, we computed the cohomology of $(\Lambda^{*,0} \otimes \mathcal{C}_-, d_2 + D)$ on six-dimensional Calabi-Yau manifolds.

On manifolds which admit a connection with holonomy $Sp(k)$, there are $k + 1$ differential spin operators associated to $k + 1$ parallel spinors. Two of these are the same as those of the $SU(2k)$ manifolds investigated above. We shall not present a full analysis in this case but we shall express a third spin differential operator on hyperKähler manifolds in terms of a Dolbeault operator.

On manifolds which admit a connection with holonomy $Spin(7)$, there is one differential spin operator d associated to one parallel spinor and a real complex $(\mathcal{C}_{\mathbb{R}}, d)$. In addition, $d^2 = 0$ provided the connection ∇ is the Levi-Civita connection of a $Spin(7)$ metric. The spin cohomology is isomorphic to de Rham cohomology.

On manifolds which admit a connection with holonomy G_2, there is again one differential spin operator d and a real complex $(\mathcal{C}_{\mathbb{R}}, d)$. In addition, $d^2 = 0$ provided the connection ∇ is the Levi-Civita connection of a G_2 metric. The spin cohomology of $(\mathcal{C}_{\mathbb{R}}, d)$ is isomorhic to two copies of the de Rham cohomology relatively shifted by one degree.

This paper has been organized as follows: in Section 2, we summarize the properties of Clifford algebras and spin representations which we use later. In Section 3, we explore the general properties of the linear differential operators (1.1), define the twisted complexes and present the algebraic cohomology operators. In Section 4, we investigate the properties of complex spin cohomology and derive the conditions for $d^2 = 0$. In addition we compute the Laplace operators. In Section 5, we investigate various kinds of twisted complex cohomology. In Section 6, we relate the complex spin cohomology to the Dolbeault cohomology for spin and $spin_c$ manifolds. In Section 7, we compute the complex spin cohomology and a twisted spin cohomology on a six-dimensional Calabi-Yau manifold. In Section 8, we investigate the spin cohomology of manifolds that admit a connection with holonomy contained in $Sp(k)$. In Section 9, we explore the properties of some real spin cohomologies. In Sections 10 and 11, we investigate the spin cohomology of manifolds that admit a connection with holonomy $Spin(7)$ and G_2, respectively.

2. Preliminaries

The investigation of spin cohomology involves a detailed description of spinor representations. Because of this and to establish notation, we shall review some aspects of spinor representations in various dimensions [3,5]. We shall focus on the manifolds with Euclidean signature but the analysis can be easily extended to other signatures.

Let $V = \mathbb{R}^n$ be a real vector space equipped with the standard Euclidean inner product. If $n = 2m$ even, the complex spin (Dirac) representation of $Spin(2m)$, $\Delta = \Delta(V)$, is reducible and decomposes to two irreducible representations, $\Delta = \Delta^+ \oplus \Delta^-$. To construct these spin representations let $e_1, \ldots, e_n$ be an orthonormal basis in $\mathbb{R}^n$, $n = 2m$, and J be a complex structure in V, $J(e_i) = e_{i+m}$. We identify V and its dual using the Euclidean inner product. Next consider the subspace $U = \mathbb{R}^m$ generated by $e_1, \ldots, e_m$. Clearly $V = U \oplus J(U)$. The Euclidean inner product on V can be extended to a hermitian inner product in $V_{\mathbb{C}} = V \otimes \mathbb{C}$ denoted by $\langle , \rangle$, i.e.

$$\langle z^\mu e_\mu, w^\nu e_\nu \rangle = \sum_\mu \bar{z}^\mu w^\mu, \tag{2.1}$$

where $\bar{z}$ is the standard complex conjugate of z in $V_{\mathbb{C}}$. The space of spinors $\Delta(V) = \Lambda^*(U_{\mathbb{C}})$, where $U_{\mathbb{C}} = U \otimes \mathbb{C}$. In addition, $\Delta^+ = \Lambda^{\text{even}}(U_{\mathbb{C}})$ and $\Delta^- = \Lambda^{\text{odd}} U_{\mathbb{C}}$. The spinors in Δ^+ are called chiral while those in Δ^- anti-chiral. The inner product (2.1) can be easily extended to Δ and it is called the Dirac inner product on the space of spinors. The generators of the Clifford algebra e_μ are represented on Δ as

$$\begin{aligned} \Gamma(e_i)\eta &= e_i \cdot \eta = e_i \wedge \eta + e_i \lrcorner \eta, \quad i \le m \\ \Gamma(e_{m+i})\eta &= e_{i+m} \cdot \eta = -ie_i \wedge \eta + ie_i \lrcorner \eta, \quad i \le m, \end{aligned} \tag{2.2}$$

where $e_i \lrcorner$ is the adjoint of $e_i \wedge$ with respect to $\langle , \rangle$. It is convenient to denote the generators $\Gamma(e_\mu) = \Gamma_\mu$ and they are often called gamma matrices. Clearly $\Gamma_\mu : \Delta^\pm \to \Delta^\mp$. The linear maps Γ_μ are hermitian with respect to the inner product $\langle , \rangle$, $\langle \Gamma_\mu \eta, \theta \rangle = \langle \eta, \Gamma_\mu \theta \rangle$, and satisfy the Clifford algebra relations $e_\mu e_\nu + e_\nu e_\mu = \Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 0$, for $\mu \ne \nu$, $(e_\mu)^2 = (\Gamma_\mu)^2 = 1$.

Next define the maps $A = \Gamma_1 \Gamma_2 \ldots \Gamma_m$ and $B = \Gamma_{m+1} \ldots \Gamma_n$ and the inner products on Δ as

$$\begin{aligned} A(\eta, \theta) &= \langle A(\bar{\eta}), \theta \rangle \\ B(\eta, \theta) &= \langle B(\bar{\eta}), \theta \rangle, \end{aligned} \tag{2.3}$$

which we denote with the same symbol, where $\bar{\eta}$ is the standard complex conjugate of η in $\Lambda^*(V_{\mathbb{C}})$ The inner products A, B are sometimes also called charge conjugation matrices. These have the following properties:

$$\begin{aligned} A(\eta, \theta) &= (-1)^{(1/2)m(m-1)} A(\theta, \eta) \\ B(\eta, \theta) &= (-1)^{(1/2)m(m+1)} B(\theta, \eta). \end{aligned} \tag{2.4}$$

Therefore, A (B) is symmetric for $m = 4k, 4k+1$ $(m = 4k, 4k+3)$ and skew-symmetric for $m = 4k+2, 4k+3$ $(m = 4k+2, 4k+1)$. In addition, we have

$$\begin{aligned} A(\Gamma_\mu \eta, \theta) &= (-1)^{m-1} A(\eta, \Gamma_\mu \theta), \quad 1 \le \mu \le n \\ B(\Gamma_\mu \eta, \theta) &= (-1)^m B(\eta, \Gamma_\mu \theta), \quad 1 \le \mu \le n \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} A(\Gamma_\mu \eta, \Gamma_\mu \theta) &= (-1)^{m-1} A(\eta, \theta), \quad 1 \le \mu \le n \\ B(\Gamma_\mu \eta, \Gamma_\mu \theta) &= (-1)^m B(\eta, \theta), \quad 1 \le \mu \le n. \end{aligned} \tag{2.6}$$

Therefore, A is $Pin(2m)$ invariant for $m = 4k+1, 4k+3$ while B is $Pin(2m)$ invariant for $m = 4k, 4k+2$. Both A, B are $Spin(n)$-invariant. A consequence of the above relations is

$$\begin{aligned} A(\eta, \Gamma_\mu \theta) &= (-1)^{(1/2)(m-1)(m+2)} A(\theta, \Gamma_\mu \eta) \\ B(\eta, \Gamma_\mu \theta) &= (-1)^{(1/2)m(m+3)} B(\theta, \Gamma_\mu \eta) \end{aligned} \tag{2.7}$$

Therefore, the gamma-matrices are symmetric with respect to the inner product $A(B)$ for $m = 4k+1, 4k+2$ $(m = 4k, 4k+1)$ while they will be skew-symmetric for $m = 4k, 4k+3$ $(m = 4k+2, 4k+3)$.

Because of the existence of invariant non-degenerate inner products the dual of Δ^* can be identified with Δ. To make this identification precise, let us denote with C either A or B. Given a basis $\{\epsilon_A; A = 1, \ldots, \dim \Delta\}$ in Δ let us denote with $\{\epsilon^A; A = 1 \ldots \dim \Delta\}$ the dual basis in Δ^*, $\epsilon^B(\epsilon_A) = \Delta^B_A$. The inner product C^{-1} in Δ^* induced by C is

$$\sum_E C^{-1}(\epsilon^E, \epsilon^A)C(\epsilon_E, \epsilon_B) = \Delta^A_B \tag{2.8}$$

The co-spinor $C(\eta)$ associated with the spinor η under the isomorphism C is defined as $C(\eta)(\theta) = C(\theta, \eta)$, $\eta, \theta \in \Delta$, i.e. in the above basis $\eta_A = C_{AB}\eta^B$. The inverse transformation C^{-1} is defined as $C^{-1}(\psi)(\chi) = C^{-1}(\psi, \chi)$, $\psi, \chi \in \Delta^*$, i.e. $\psi^A = \psi_B(C^{-1})^{BA}$. Notice that the maps $A, B : \Delta^\pm \to \Delta^\pm$ for $m = 4k, 4k+2$ while $A, B : \Delta^\pm \to \Delta^\mp$ for $m = 4k+1, 4k+3$. Therefore, in the former case the dual $\Delta_\pm$ of $\Delta^\pm$ under A, B is identified with $\Delta^\pm$, $\Delta_\pm = \Delta^\pm$ while in the latter the dual $\Delta_\pm$ of $\Delta^\pm$ is identified with $\Delta^\mp$, $\Delta_\pm = \Delta^\mp$.

There are two ways to construct the spin representation Δ in odd dimensions. One is to write $V' = \mathbb{R}^{2m+1} = V \oplus \mathbb{R}\langle e_{2m+1}\rangle$ and extend the Euclidean inner product (2.1) from V to V', $\langle e_{2m+1}, e_{2m+1}\rangle = 1$, $\langle V, e_{2m+1}\rangle = 0$. The gamma matrices Γ_μ, $1 \le \mu \le 2m$, are defined as in the even-dimensional case and

$$\Gamma_{2m+1} = i^m \Gamma_1 \ldots \Gamma_{2m}. \tag{2.9}$$

The $Spin(2m+1)$ spin representation, Δ, is $\Delta = \Delta^+ \oplus \Delta^-$, where Δ^+, Δ^- are the $Spin(2m)$ spin representations. (There are no chiral spinors in odd dimensions.) The invariant inner product on $Spin(2m+1)$ representation Δ is the $Pin(2m)$ invariant inner product on $\Delta^+ \oplus \Delta^-$.

Alternatively, we take $V = U \oplus J(U)$ as for $n = 2m$ and write $U = U_0 \oplus \mathbb{R}\langle e_{2m}\rangle$. Then $V_0 = U \oplus J(U_0)$ has dimension $2m-1$. The gamma matrices are $\tilde{\Gamma}_\mu = i\Gamma_\mu\Gamma_{2m}$, $1 \le \mu \le 2m-1$, where Γ_μ are the gamma matrices of the $Spin(2m)$ spin representation. These induce a representation of $Pin(2m-1)$ onto the $\Delta^\pm$ representations of $Spin(2m)$.

For later convenience, we introduce the notation $(C\Gamma_\mu)(\eta, \theta) = C(\eta, \Gamma_\mu\theta)$ and similarly $(C\Gamma_{\mu_1\ldots\mu_p})(\eta, \theta) = C(\eta, \Gamma_{\mu_1\ldots\mu_p}\theta)$, where

$$\Gamma_{\mu_1\ldots\mu_p} = \frac{1}{p!}\sum_\sigma (-1)^{|\sigma|}\Gamma_{\mu_{\sigma(1)}}\ldots\Gamma_{\mu_{\sigma(p)}} \tag{2.10}$$

and σ is a permutation. The symmetry of the inner product and that of the gamma matrices can be re-expressed as $C(\eta, \theta) = (-1)^{s_C}C(\theta, \eta)$, where $s_C = 0$ if C is symmetric and $s_C = 1$ if C is skew-symmetric, and similarly $C\Gamma_\mu(\eta, \theta) = (-1)^{s_\Gamma}C\Gamma_\mu(\theta, \eta)$, where $s_\Gamma = 0$ if $C\Gamma_\mu$ is symmetric and $s_\Gamma = -1$ if $C\Gamma_\mu$ is skew-symmetric. From these one can also find that

$$C\Gamma_{\mu_1\ldots\mu_p}(\eta, \theta) = (-1)^{(1/2)p(p-1)}(-1)^{(p+1)s_C + ps_\Gamma}C\Gamma_{\mu_1\ldots\mu_p}(\theta, \eta). \tag{2.11}$$

Similarly, we define $(\Gamma_{\mu_1\ldots\mu_p}C^{-1})(\psi, \chi) = C^{-1}(\Gamma_{\mu_1\ldots\mu_p}\psi, \chi)$, where $\chi, \psi \in \Delta^*$.

The product of two co-spinor representations can be decomposed in terms of forms as $\Delta^* \otimes \Delta^* = \sum_{p=1}^n \Lambda^p(V) \otimes \mathbb{C}$. In particular, one can write

$$\begin{aligned}&(\psi \otimes \chi)(\eta \otimes \theta)\\ &= \frac{1}{\dim \Delta_n}\left(C^{-1}(\psi, \chi)C(\eta, \theta) + \sum_{p=1}^n \frac{(-1)^{p(s_\Gamma + s_C)}}{p!}(\Gamma^{\mu_1\ldots\mu_p}C^{-1})(\psi, \chi)C\Gamma_{\mu_1\ldots\mu_p}(\eta, \theta)\right),\end{aligned} \tag{2.12}$$

where $\eta, \theta \in \Delta$ and $\psi, \chi \in \Delta^*$. The above decomposition is valid after restricting to $\Delta_\pm$ co-spinor representations and to real co-spinor representations. We shall state the formulae later. The formula (2.12) is also known as Fierz identity.

In the above formalism, it is possible to explicitly present the spinors that are invariant under the action of certain subgroups of $Spin(n)$. We shall mainly focus on the subgroups $G \subset Spin(n)$ which arise as special holonomy groups in the Berger classification and the associated manifolds admit a parallel spinor. These spinors have been given in [3]. Here, we shall summarize the results and adjust the various formulae because of differences in the conventions.

(i) $G = SU(m) \subset Spin(2m)$. The invariant spinors under the $SU(m) \subset Spin(2n)$ are $1, e_1 \wedge e_2 \wedge \ldots \wedge e_m$. This can be easily seen by decomposing the $Spin(2m)$ representations $\Delta^\pm$ under $SU(m)$. If $m = 4k, 4k+2$ both invariant spinors are of the same chirality, i.e. they are elements of Δ^+ while if $m = 4k+1, 4k+3$, they have opposite chiralities. In addition observe that $\Gamma_j - i\Gamma_{m+j}(1) = 0$ and $\Gamma_j + i\Gamma_{m+j}(e_1 \wedge \ldots e_m) = 0$, $j = 1, \ldots, m$. Therefore, the invariant spinors are pure spinors with respect to the holomorphic and antiholomorphic parts of the decomposition of $V \otimes \mathbb{C}$ with respect to the complex structure J.

(ii) $G = Sp(k) \subset Spin(4k)$. The invariant spinors are $1, e_1 \wedge e_2 \wedge, \ldots, \wedge e_{2k}, \omega, \omega^2, \ldots, \omega^{k-1}$ where $\omega = e_1 \wedge e_2 + \cdots + e_{2k-1} \wedge e_{2k}$ which is the symplectic form in $U \subset \Delta^+$. Therefore, there are $k+1$ parallel spinors.

(iii) $G = Spin(7) \subset Spin(8)$. The invariant spinor is $\frac{1}{\sqrt{2}}(e_1 - e_2 \wedge e_3 \wedge e_4)$.

(iv) $G = G_2 \subset Spin(7)$. The invariant spinor is $\frac{1}{\sqrt{2}}(e_1 - e_2 \wedge e_3 \wedge e_4)$.

3. Differential and algebraic operations on spinors

3.1. First order differential operators

Let M be a spin manifold equipped with a spin connection ∇ which admits a parallel spinor ζ, $\nabla\zeta = 0$. We shall focus on even-dimensional manifolds. Some of the results can be easily extended to the odd dimensional case. We define $\mathcal{C}_\pm = \Lambda^*(\Delta_\pm)$ and $\mathcal{C} = \Lambda^*(\Delta^*)$ equipped with the wedge product $\bar{\wedge}$.

Definition 1. The spin operator d is a linear differential operator $d : \mathcal{C}(M) \to \mathcal{C}(M)$, and similarly $d : \mathcal{C}_\pm(M) \to \mathcal{C}_\pm(M)$, such that

$$d\phi = \sum_{\mu=1}^{n} C^\mu_\zeta \bar{\wedge} \nabla_\mu \phi, \tag{3.1}$$

where $C^\mu_\zeta(\eta) = C\Gamma^\mu(\zeta, \eta)$.

Clearly $d : \mathcal{C}^\ell \to \mathcal{C}^{\ell+1}$ and $d : \mathcal{C}^\ell_\pm \to \mathcal{C}^{\ell+1}_\pm$. Choosing a basis in the space of co-spinors $\{\epsilon^A : A = 1, \ldots, 2^m\}$, the d operator can be written as

$$\begin{aligned} d\phi_{A_1,A_2\ldots A_{\ell+1}} &= \zeta^B C\Gamma^\mu_{BA_1} \nabla_\mu \phi_{A_2\ldots A_{\ell+1}} \\ &\quad + \text{cyclic}(A_1, A_2, \ldots, A_{\ell+1}) B, A_1, \ldots, A_{\ell+1} = 1, \ldots, \dim \Delta. \end{aligned} \tag{3.2}$$

The operator d depends on the choice of parallel spinor ζ and the connection ∇. Although subject to the data above, the operator d can always be defined on $\mathcal{C}(M)$, the restriction on d onto $\mathcal{C}_+$ or $\mathcal{C}_-$ depends on the choice of the parallel spinor ζ. Since this depends on the dimension of the manifold and the choice of the parallel spinor, we shall explain the general properties of the operator d acting on $\mathcal{C}$, and later we shall specialize into the various cases.

Evaluating d^2, we find

$$d^2\phi = \frac{1}{2} C^\mu_\zeta \bar{\wedge} C^\nu_\zeta \bar{\wedge} R_{\mu\nu}\phi, \tag{3.3}$$

where R is the curvature of the connection ∇. Under certain conditions the operator d can be nilpotent, $d^2 = 0$. This depend on the choice of the spinor ζ and the connection ∇. There are two large classes of examples for which $d^2 = 0$.

- Group manifolds equipped with the left or the right invariant connections.
- Manifolds that admit a pure parallel spinor.

In the case of group manifolds $R = 0$. Therefore, d operators associated with the left- or right-invariant connections are all nilpotent. The different d operators that can be constructed on group manifolds are determined by the orbits of $Spin(n)$ in Δ.

A spinor is pure if the subspace

$$W(\zeta) = \{v \in V_{\mathbb{C}}, v_\mu \Gamma^\mu \zeta = 0\} \tag{3.4}$$

of $V_{\mathbb{C}}$ has dimension $\frac{1}{2}\dim_{\mathbb{C}}(V_{\mathbb{C}})$. We can use the inner product to decompose $V_{\mathbb{C}} = W(\zeta) \oplus Z$.

Proposition 1. *The operator d is nilpotent if, in addition to $\nabla\zeta = 0$, the curvature R vanishes along the subspace $\Lambda^2(Z) \subset \Lambda^2(V_{\mathbb{C}})$, i.e.*

$$R|_{\Lambda^2(Z)} = 0. \tag{3.5}$$

Proof. The curvature R can be viewed as a map from $R : \Lambda^2(M) \to \Lambda^2(M)$. Therefore,

$$d^2\phi = \frac{1}{2} C^\mu_\zeta \bar{\wedge} C^\nu_\zeta \bar{\wedge} R_{\mu\nu}\phi = \frac{1}{2} [C^\mu_\zeta \bar{\wedge} C^\nu_\zeta \bar{\wedge} R_{\mu\nu}]|_{\Lambda^2(Z)}\phi = 0. \quad \square \tag{3.6}$$

Clearly, this condition can be generalized to spinors ζ which are not pure but $W(\zeta) \neq \emptyset$.

The conditions on the curvature required for $d^2 = 0$ can also be determined using (2.12). In particular, we have

Proposition 2. *The conditions on the curvature R for $d^2 = 0$ can be expressed in terms of the forms associated with the parallel spinor ζ.*

Proof. We compute d^2 using (2.12) to find

$$\begin{aligned} d^2\phi &= \frac{1}{2\dim \Delta_n} \left(\sum_{p=0}^{n} \frac{(-1)^{p(s_\Gamma+s_C)}}{p!} (\Gamma^{\rho_1\ldots\rho_p} C^{-1})(C^\mu_\zeta, C^\nu_\zeta) \right) C\Gamma_{\rho_1\ldots\rho_p} \bar{\wedge} R_{\mu\nu}\phi \\ &= \frac{1}{2\dim \Delta_n} \left(\sum_{p=0}^{n} \frac{(-1)^{(p+1)(s_\Gamma+s_C)}}{p!} C(\zeta, \Gamma^\mu \Gamma^{\rho_1\ldots\rho_p} \Gamma^\nu \zeta) \right) C\Gamma_{\rho_1\ldots\rho_p} \bar{\wedge} R_{\mu\nu}\phi \\ &= \frac{1}{2\dim \Delta_n} \\ &\quad \times \left(\sum_{p=0}^{n} \frac{(-1)^{(p+1)(s_\Gamma+s_C)}}{p!} [C(\zeta, \Gamma^{\mu\rho_1\ldots\rho_p\nu}\zeta) + p(p-1) g^{\mu\rho_1} C(\zeta, \Gamma^{\rho_2\ldots\rho_{p-1}}\zeta) g^{\rho_p\nu}] \right) \\ &\quad \times C\Gamma_{\rho_1\ldots\rho_p} \bar{\wedge} R_{\mu\nu}\phi, \end{aligned} \tag{3.7}$$

where g is the metric on the manifold. In the above sum over p only the terms with $C\Gamma_{\rho_1\ldots\rho_p}$ skew-symmetric contribute. Sufficient conditions on the curvature for $d^2 = 0$ are

$$[C(\zeta, \Gamma^{\mu\rho_1\ldots\rho_p\nu}\zeta) + p(p-1)C(\zeta, \Gamma^{\rho_2\ldots\rho_{p-1}}\zeta)g^{\mu\rho_1}g^{\nu\rho_p}]R_{\mu\nu} = 0 \tag{3.8}$$

for $\frac{1}{2}p(p-1) + (p+1)s_C + ps_\Gamma \in 2\mathbb{Z} + 1$. In some cases, these conditions are also necessary. □

Provided that the conditions for $d^2 = 0$ are met, we can define a cohomology theory associated with the linear differential operator d.

Definition 2. The spin cohomology, $H_d(\mathcal{C})$, is that of the graded complex $(\mathcal{C}, d)$, $d^2 = 0$, where d is as in (3.1). Similarly, the spin cohomology, $H_d(\mathcal{C}_\pm)$, is that of the graded complex $(\mathcal{C}_\pm, d)$.

3.2. *Twisted complexes*

There are several ways to twist the complexes $\mathcal{C}$ and $\mathcal{C}_\pm$. Here, we shall consider two cases which we shall describe below.

3.2.1. *The complexes $\mathcal{C} \otimes E$ and $\mathcal{C}_\pm \otimes E$*

Let E be a vector bundle E over the spin manifold M equipped with a connection ∇^E. One way to twist the complexes $\mathcal{C}$ and $\mathcal{C}_\pm$ is to consider $\mathcal{C} \otimes E$ and $\mathcal{C}_\pm \otimes E$. Let ζ be a parallel spinor with respect to a spin connection ∇^M on the manifold M induced from the tangent bundle, $\nabla^M\zeta = 0$. The spin differential operator d is

$$d\phi = C^\mu_\zeta \bar{\wedge} \nabla_\mu \phi, \tag{3.9}$$

where $\nabla = \nabla^M \otimes 1 + 1 \otimes \nabla^E$ on $\mathcal{C} \otimes E$ or on $\mathcal{C}_\pm \otimes E$ and $\phi \in \mathcal{C} \otimes E$ or $\mathcal{C}_\pm \otimes E$, respectively.

The condition $d^2 = 0$ implies conditions on both the curvature R of M and the curvature F of the connection ∇^E of the bundle E.

Theorem 1. *The operator d is nilpotent providing that both the curvature R of the manifold M and the curvature R of E satisfy either (3.5) or (3.7).*

Proof. This is similar to the proof given in the previous section. □

There is a particular twisted complex of the this type that we shall consider by taking $E = \Lambda^*(M)$ or $E = \Lambda^*(M) \otimes \mathbb{C}$. We shall see that in this case one can define certain algebraic operators with are nilpotent. The spin cohomology, $H_d(\mathcal{C} \otimes E)$, of the linear operator d for the twisted complex $(\mathcal{C} \otimes E, d)$, can be defined in analogy with the spin cohomology of the untwisted case in the previous section. This definition can be extended for $H_d(\mathcal{C}_\pm \otimes E)$.

3.2.2. *The complexes $\mathcal{C}(E) = \Lambda^*(\Delta^* \otimes E)$ and $\mathcal{C}(E)_\pm = \Lambda^*(\Delta_\pm \otimes E)$*

These complexes allow the definition of the spin operator d operator on manifolds that do not admit a spin structure but admit a *spin$_c$* or in general a *Spin$_G$* structure. Another use of twisted complexes $\mathcal{C}(E)$ is that they allow the imposition of a reality condition. It is known that there are not real (Majorana) spin representations for $n = 8k + 4$ dimensional manifolds and so there is not a real complex $\mathcal{C}$. However, it is possible to construct a real complex $\mathcal{C}(E)$ by taking E to be a rank two $SU(2)$ bundle.

Let ζ be a parallel section of $\Delta \otimes E$ with respect to a connection $\mathcal{D} = \nabla^M \otimes 1 + 1 \otimes \nabla^E$, where ∇^M is a spin connection on the manifold M induced from the tangent bundle and ∇^E is a connection on the vector bundle E. The operator d on $\mathcal{C}(E)$ or $\mathcal{C}(E)_\pm$ is defined as

$$d\phi = C_\zeta^\mu \bar{\wedge} \mathcal{D}_\mu \phi, \tag{3.10}$$

where $\bar{\wedge}$ is the wedge operation in $\Lambda^*(\Delta^* \otimes E)$. One consequence of this definition is that ζ is not necessary a parallel section of the spinor bundle but of $\Delta \otimes E$.

The case that it is of most interest to us is that for which E is a line bundle. In this case, the conditions for $d^2 = 0$ can be expressed as conditions on the curvature R and F of the manifold and of the line bundle, respectively. The formulae are similar to those in (3.5) and (3.7).

The construction can be further generalized in the case for which there is no a spin structure but there is a $spin_c$ structure. In this case, although the spin bundle Δ is not well-defined $\Delta \otimes E$ is and so is $\mathcal{C}(E)$.

Provided that the conditions for $d^2 = 0$ are met, we can define the twisted spin cohomology, $H_d(\mathcal{C}(E))$, of the graded complex $(\mathcal{C}(E), d)$. Similary we can define the twisted spin cohomology, $H_d(\mathcal{C}_\pm(E))$, of the graded complex $(\mathcal{C}_\pm(E), d)$

3.3. Algebraic operations

3.3.1. The algebraic operator $D_{(p)}$

There are several algebraic cohomology operations that can be defined on the twisted complexes $\Lambda^* \otimes \mathcal{C}$, $\Lambda^* \otimes \mathcal{C}_\pm$, $Sym^* \otimes \mathcal{C}$ and $Sym^* \otimes \mathcal{C}_\pm$, where $Sym^* = \oplus_{p=0}^\infty Sym^p$ and Sym^p is the symmetrized product of p copies of Λ^1.

The maps $C\Gamma^{(p)} : \Delta \otimes \Delta \to \Lambda^p$

$$C\Gamma^{(p)}(\eta, \theta) = \frac{1}{p!} C\Gamma_{\mu_1 \dots \mu_p}(\eta, \theta) e^{\mu_1} \wedge \dots \wedge e^{\mu_p} \tag{3.11}$$

are skew-symmetric, i.e. $C\Gamma^{(p)}(\eta, \theta) = -C\Gamma^{(p)}(\theta, \eta)$, provided that $\frac{1}{2}p(p-1) + (p+1)s_C + ps_\Gamma \in 2\mathbb{Z} + 1$ as it can been seen from (2.11).

Definition 3. The algebraic spin operator $D_{(p)} : \Lambda^q(M) \otimes \mathcal{C}^\ell(M) \to \Lambda^{q-p}(M) \otimes \mathcal{C}^{\ell+2}(M)$ is

$$D_{(p)}\phi = \frac{(-1)^{(1/2)p(p-1)+\ell}}{2(q-p)!p!\ell!} \times (C\Gamma^{\mu_1 \dots \mu_p})_{A_1 A_2} \phi_{\mu_1 \dots \mu_p \nu_1 \dots \nu_{q-p} A_3, \dots, A_{\ell+2}} e^{\nu_1} \wedge \dots \wedge e^{\nu_{q-p}} \otimes \epsilon^{A_1} \bar{\wedge} \dots \bar{\wedge} \epsilon^{A_{\ell+2}}, \tag{3.12}$$

if $q \geq p$ and $D_{(p)} = 0$ for $p > q$, where $C\Gamma^{(p)}$ is skew-symmetric.

It is straightforward to show that

Proposition 3. *$D_{(p)}$ is nilpotent, $D_{(p)}^2 = 0$, provided that $p \in 2\mathbb{Z} + 1$.*

Proposition 4. *The algebraic spin operator $D_{(p)}$ can be restricted on $\Lambda^* \otimes \mathcal{C}_\pm$, iff* $\dim M = 8k + 2, 8k + 6$.

Proof. It can be seen from the properties of spinor inner product C summarized in Section 2 that for $\dim M = 8k + 2, 8k + 6$, $C\Gamma^{(p)} : \Delta_\pm \otimes \Delta_\pm \to \Lambda^p$, $p \in 2\mathbb{Z} + 1$. $\square$

In what follows, when we refer to the algebraic spin operator on $\Lambda^* \otimes \mathcal{C}_\pm$ complexes we shall assume the condition of the above proposition applies and $\dim M = 8k+2, 8k+6$.

The algebraic spin operator $D_{(p)}$ can be extended to twisted complexes $\Lambda^* \otimes \mathcal{C} \otimes E$ and $\Lambda^* \otimes \mathcal{C}_\pm \otimes E$ in a straightforward way. There is also an extension to the twisted complexes $\Lambda^* \otimes \mathcal{C}(E)$ and $\Lambda^* \otimes \mathcal{C}_\pm(E)$ provided that E is equipped with and inner product h. In particular, we define

$$D_{(p)}: \quad \Lambda^q(M) \otimes \mathcal{C}^\ell(E) \to \Lambda^{q-p}(M) \otimes \mathcal{C}^{\ell+2}(E) \tag{3.13}$$

where

$$D_{(p)}\phi = \frac{(-1)^{(1/2)p(p-1)+\ell}}{2(q-p)!p!\ell!}(C\Gamma^{\mu_1\ldots\mu_p} \otimes h)_{I_1A_1,I_2A_2}\phi_{\mu_1\ldots\mu_p\nu_1\ldots\nu_{q-p}I_3A_3\ldots I_{\ell+2}A_{\ell+2}}$$
$$e^{\nu_1} \wedge \ldots \wedge e^{\nu_{q-p}} \otimes \epsilon^{I_1A_1} \bar{\wedge} \ldots \bar{\wedge} \epsilon^{I_{\ell+2}A_{\ell+2}}, \tag{3.14}$$

if $q \geq p$ and $D_{(p)} = 0$ for $p > q$. This operation is well defined provided that $C\Gamma^{(p)} \otimes h$ is skew-symmetric. This is the case when either $C\Gamma^{(p)}$ is symmetric and h is skew-symmetric or $C\Gamma^{(p)}$ is skew-symmetric and h is symmetric. In all the above cases $D^2_{(p)} = 0$ provided $p \in 2\mathbb{Z}+1$.

Definition 4. The algebraic spin cohomology $H_{D_{(p)}}(\Lambda^* \otimes \mathcal{C})$ is defined as the cohomology of the double graded complex $(\Lambda^* \otimes \mathcal{C}, D_{(p)})$. This definition can be extended to the rest of the twisted and untwisted spin complexes.

A particular case of this operation is for $p = 1$. In this case, we have $D = D_{(1)} : \Lambda^* \otimes \mathcal{C} \to \Lambda^* \otimes \mathcal{C}$, where

$$D\phi = \frac{(-1)^\ell}{2(q-1)!\ell!}(C\Gamma^\mu)_{A_1A_2}\phi_{\mu_1\nu_1\ldots\nu_{q-1}A_3\ldots A_{\ell+2}}e^{\nu_1} \wedge \ldots \wedge e^{\nu_{q-1}} \otimes \epsilon^{A_1} \bar{\wedge} \ldots \bar{\wedge} \epsilon^{A_{\ell+2}}, \tag{3.15}$$

if $q \geq 1$ and $D_{(p)} = 0$ for $q = 0$. In particular, D is defined on $\Lambda^* \otimes \mathcal{C}$ and $\Lambda^* \otimes \mathcal{C} \otimes E$ for $m = 4k, 4k+2$ if $C = A$ and for $m = 4k+3$ if $C = B$. It is also defined on $\Lambda^* \otimes \mathcal{C}_\pm$ and $\Lambda^* \otimes \mathcal{C}_\pm \otimes E$ for $m = 4k+2$ if $C = A$ and for $m = 4k+3$ if $C = B$, $\dim M = 2m$. In the twisted case $\Lambda^* \otimes \mathcal{C}(E)$, the operator D can be defined in all the cases for which E admits a fibre inner product such that $C\Gamma^{(1)} \otimes h$ is skew-symmetric.

Proposition 5. *$D_{(p)}$ anti-commutes with the differential operator d, i.e.*

$$dD_{(p)} + D_{(p)}d = 0. \tag{3.16}$$

Proof. We can show this after a direct computation using the property of the connection ∇^M of the manifold to be a spin connection induced from the tangent bundle. In the twisted case (3.14) this also the case provided that $\nabla^E h = 0$. □

The differential and algebraic spin cohomology operators on the various untwisted and twisted complexes above can be combined into an new spin cohomology operator $d + D$. The new cohomology operator $d + D$ defines a new cohomology, H_{d+D}, which can be computed using spectral sequences. We shall describe such computation on Calabi-Yau manifolds of dimension six.

3.3.2. The algebraic operator $\hat{D}$

Apart from the $D_{(p)}$ algebraic operator, there is another algebraic operation $\hat{D}$.

Definition 5. The algebraic operator $\hat{D}$ on the complex $Sym^* \otimes \mathcal{C}$ is

$$\hat{D}: \quad Sym^q(M) \otimes \mathcal{C}^\ell \to Sym^{q-1}(M) \otimes \mathcal{C}^{\ell+1}, \tag{3.17}$$

where

$$\hat{D}\phi = (-1)^\ell \frac{1}{(q-1)!\ell!} (C_\zeta)^\mu_{A_1} \phi_{\mu\nu_1,\dots\nu_{q-1}A_2\dots A_{\ell+1}} e^{\nu_1} \wedge \dots \wedge e^{\nu_{q-1}} \otimes \epsilon^{A_1} \bar{\wedge} \dots \bar{\wedge} \epsilon^{A_{\ell+1}} \tag{3.18}$$

if $q \geq 1$ and $\hat{D} = 0$ if $q = 0$.

It is straightforward to extend this definition to the other untwisted and twisted complexes. Moreover, one can show that

Proposition 6. *$\hat{D}$ is nilpotent, $\hat{D}^2 = 0$.*

As in the case of $D_{(p)}$ algebraic spin operator

Proposition 7. *$\hat{D}$ anti-commutes with the differential operators d, i.e.*

$$d\hat{D} + \hat{D}d = 0. \tag{3.19}$$

A consequence of this is that one can define a new cohomology operator $d + \hat{D}$ and an associated cohomology $H_{d+\hat{D}}$ which can be computed using a spectral sequence.

Proposition 8. *Let $\hat{D}$ be the spin algebraic operator on $Sym^* \otimes \mathcal{C}$ defined as in (3.18). If C_ζ is an isomorphism, then $H^*_{\hat{D}} = \mathcal{C}$.*

Proof. Since C_ζ is an isomorphism, then $Sym^* \otimes \mathcal{C} = Sym^* \otimes \Lambda^*$. $Sym^p \otimes \Lambda^q$ can be decomposed under $GL(n, \mathbb{C})$ into two irreducible representations. These have dimensions

$$\begin{aligned} \Delta_1 &= \tfrac{n(n+1)\dots(n+p-1)(n-1)\dots(n-q)}{(p+q)(p-1)!q!} \\ \Delta_2 &= \tfrac{n(n+1)\dots(n+p)(n-1)\dots(n-q+1)}{(p+q)p!(q-1)!}. \end{aligned} \tag{3.20}$$

Clearly $\mathrm{Ker}\hat{D}|_{Sym^0 \times \Lambda^q} = \Lambda^q$. In addition

$$\mathrm{Ker}\hat{D}|_{Sym^p \times \Lambda^q} = \hat{D}(Sym^{p+1} \times \Lambda^{q-1}), \quad p > 0, \tag{3.21}$$

with $\dim \mathrm{Ker}\hat{D}|_{Sym^p \times \Lambda^q} = \Delta_2$ Therefore, all cohomology $H^{p,q}_{\hat{D}} = 0$ for $p > 0$ and $H^{0,q}_{\hat{D}} = \Lambda^q = \mathcal{C}^q$. □

This theorem can be thought as a consequence of the Spencer cohomology [9]. Clearly, the above result can be generalized to $Sym^* \otimes \mathcal{C}_\pm$ and twisted complexes.

4. Manifolds with connections of holonomy $SU(m)$ and spin cohomology

As we have mentioned on even-dimensional Riemannian manifolds, there are two complex spin representations $\Delta^\pm$. In addition there are real spin representations provided that $m = 4k + 1, 4k + 3, 4k$, $\dim M = 2m$. In what follows, we shall focus on the spin cohomology associated the complex representations. The spin cohomology associated with real representations will be investigated later.

4.1. Complex spinor representations

Let M be a Riemannian manifold equipped with a spin connection ∇ with $\mathrm{hol}(\nabla) \subseteq SU(m)$. We take that the metric on M to be compatible with the parallel almost complex structure J. There are two distinct ∇-parallel complex spinors. These are given by $\zeta_1 = 1$, $\zeta_2 = e_1 \wedge \ldots \wedge e_m$. These spinors are of different chirality if $m = 4k+1, 4k+3$ and of the same chirality if $m = 4k, 4k+2$. Therefore, there are two first order differential spin operators d_1 and d_2 associated with the spinors ζ_1 and ζ_2, respectively. If m is odd, $d_1 : \mathcal{C}_+ \to \mathcal{C}_+$ and $d_2 : \mathcal{C}_- \to \mathcal{C}_-$ while if m is even, $d_1, d_2 : \mathcal{C}_- \to \mathcal{C}_-$. We shall treat the two cases separately.

4.1.1. The $m = 4k+1, 4k+3$ case

Theorem 2. *The operator $d_1 : \mathcal{C}_+ \to \mathcal{C}_+$ is nilpotent, $d_1^2 = 0$, provided that the (2,0) part of the curvature R of the connection ∇ with respect to the almost complex structure J vanishes.*

Proof. For this we compute d_1^2 on $\mathcal{C}_+^\ell$ to find

$$d_1^2\phi = \frac{1}{2} C_1^\mu \bar{\wedge} C_1^\nu \bar{\wedge} R_{\mu\nu}\phi, \tag{4.1}$$

where $R_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$ is the curvature of the connection ∇. In a spinor basis $\{\epsilon^a : a = 1, \ldots, \dim \Delta_+\}$, the above expression can be written as

$$d_1^2\phi = \frac{1}{2\ell!} (C_1)^\mu_{a_1} (C_1)^\nu_{a_2} (R_{\mu\nu}\phi)_{a_3\ldots a_{\ell+2}}\, \epsilon^{a_1} \wedge \epsilon^{a_2} \wedge \epsilon^{a_3} \wedge \ldots \wedge \epsilon^{a_{\ell+2}}. \tag{4.2}$$

Observe that the product representation $\Delta_+ \otimes \Delta_+$ can be decomposed as

$$\Delta_+ \otimes \Delta_+ = \sum_{p=1}^{\frac{m-3}{2}} \Lambda^{2p+1}(V_\mathbb{C}) \oplus \Lambda^{m+}(V_\mathbb{C}). \tag{4.3}$$

In particular using (2.12), we have

$$\chi \otimes \psi(\eta \otimes \theta) = \frac{1}{\dim_\mathbb{C} \Delta_+} \sum_{p=1}^{\frac{m-3}{2}} \frac{(-1)^{(2p+1)(s_\Gamma + s_C)}}{(2p+1)!} (\Gamma^{\mu_1\ldots\mu_{2p+1}} C^{-1})(\chi, \psi) C\Gamma_{\mu_1\ldots\mu_{2p+1}}(\eta, \theta)\,. \tag{4.4}$$

The only non-vanishing form associated with ζ_1 spinor is the m-form given by

$$\epsilon_1 = \frac{1}{m!} C(\zeta_1, \Gamma_{\rho_1\ldots\rho_m}\zeta_1) e^{\rho_1} \wedge \ldots \wedge e^{\rho_m}. \tag{4.5}$$

Moreover, from section two, we have that $C\Gamma_{\rho_1\ldots\rho_m}$ is symmetric and $C\Gamma_{\rho_1\ldots\rho_{m-2}}$ is skew-symmetric. Applying the formula (4.4) for $\chi = C_1^\mu$ and $\psi = C_1^\nu$, we find that the only non-vanishing term is

$$\begin{aligned} d_1^2\phi &= \frac{1}{2} \frac{(-1)^{(m-2)(s_\Gamma + s_C)}}{(m-2)! \dim_\mathbb{C} \Delta_+} (\Gamma^{\rho_1\ldots\rho_{m-2}} C^{-1})(C_1^\mu, C_1^\nu) C\Gamma_{\rho_1\ldots\rho_{m-2}} \bar{\wedge} R_{\mu\nu}\phi \\ &= \frac{1}{2} \frac{(-1)^{(m-1)(s_\Gamma + s_C)}}{(m-2)! \dim_\mathbb{C} \Delta_+} C(1, \Gamma^{\mu\rho_1\ldots\rho_{m-2}\nu} 1) C\Gamma_{\rho_1\ldots\rho_{m-2}} \bar{\wedge} R_{\mu\nu}\phi. \end{aligned} \tag{4.6}$$

Therefore, $d_1^2 = 0$, if the (2,0) component of the curvature R vanishes since $(\Gamma_j - i\Gamma_{j+m})1 = 0$ and so ϵ_1 is an $(0, m)$ form. □

Theorem 3. *The operator $d_2 : \mathcal{C}_- \to \mathcal{C}_-$ is nilpotent, $d_2^2 = 0$, provided that the (0,2) part of the curvature R of the connection ∇ with respect to the almost complex structure J vanishes.*

Proof. The proof of this is similar to the one presented above for the case of the ζ_1 parallel spinor. One difference is the decomposition

$$\Delta_- \otimes \Delta_- = \sum_{p=1}^{\frac{m-3}{2}} \Lambda^{2p+1}(V_{\mathbb{C}}) \oplus \Lambda^{m-}(V_{\mathbb{C}}). \tag{4.7}$$

A direct computation reveals that

$$\begin{aligned} d_2^2\phi &= \frac{1}{2}\frac{(-1)^{(m-2)(s_\Gamma+s_C)}}{(m-2)!\dim_{\mathbb{C}}\Delta_+}(\Gamma^{\rho_1\ldots\rho_{m-2}}C^{-1})(C_2^\mu, C_2^\nu)C\Gamma_{\rho_1\ldots\rho_{m-2}}\bar{\wedge}R_{\mu\nu}\phi \\ &= \frac{1}{2}\frac{(-1)^{(m-1)(s_\Gamma+s_C)}}{(m-2)!\dim_{\mathbb{C}}\Delta_+}C(e_1\wedge\ldots\wedge e_m, \Gamma^{\mu\rho_1\ldots\rho_{m-2}\nu}e_1\wedge\ldots\wedge e_m)C\Gamma_{\rho_1\ldots\rho_{m-2}}\bar{\wedge}R_{\mu\nu}\phi. \end{aligned} \tag{4.8}$$

Using $(\Gamma_j + i\Gamma_{j+m})e_1 \wedge \ldots \wedge e_m = 0$, we conclude that $d_2^2 = 0$ if the (0,2) part of the curvature R vanishes. □

Corollary 1. *The operator $d = d_1 \oplus d_2 : \mathcal{C}_+ \oplus \mathcal{C}_- \to \mathcal{C}_+ \oplus \mathcal{C}_-$ is nilpotent provided that the curvature R of the connection ∇ is (1,1) with respect to the almost complex structure J.*

Proof. It follows immediately from the two theorems above. □

We therefore conclude that there are three kinds of untwisted differential spin cohomology associated with a manifold that admits a connection with holonomy contained in $SU(m)$, $m = 4k+1, 4k+3$. The complexes are $(\mathcal{C}_+, d_1)$, $(\mathcal{C}_-, d_2)$ and $(\mathcal{C}_+ \oplus \mathcal{C}_-, d_1 \oplus d_2)$ and the associated spin cohomologies are denoted as $H_{d_1}(\mathcal{C}_+)$, $H_{d_2}(\mathcal{C}_-)$ and $H_d(\mathcal{C}_+ \oplus \mathcal{C}_-)$, respectively.

4.1.2. The $m = 4k, 4k+2$ case

In this case, both parallel spinors $\zeta_1, \zeta_2 \in \Delta^+$. Therefore, $d_1, d_2 : \mathcal{C}_- \to \mathcal{C}_-$.

Theorem 4. *The operators $d_1, d_2 : \mathcal{C}_- \to \mathcal{C}_-$ are nilpotent, $d_1^2 = 0$ and $d_2^2 = 0$, provided that the either (2,0) or the (0,2) part of the curvature R of the connection ∇ with respect to the almost complex structure J vanishes, respectively.*

Proof. The proof of this statement is similar to that for the cases $m = 4k+1, 4k+3$ described in the previous section. In particular, we have

$$\Delta_- \otimes \Delta_- = \sum_{p=0}^{\frac{m-2}{2}} \Lambda^{2p}(V_{\mathbb{C}}) \oplus \Lambda^{m-}(V_{\mathbb{C}}). \tag{4.9}$$

From the results of section two, the map $C\Gamma^{\mu_1\ldots\mu_m}$ is symmetric and $C\Gamma^{\mu_1\ldots\mu_{m-2}}$ is skew-symmetric with respect to both inner products $C = A, B$. The expressions for d_1^2 and d_2^2 are given by (4.6) and (4.8), respectively. From these, it is straightforward to see that $d_1^2 = 0$ ($d_2^2 = 0$) if the (0,2) ((2,0)) part of the curvature R of ∇ vanishes. □

Since both operators d_1, d_2 act on the same complex, one can define the operator $d = d_1 + d_2 : \mathcal{C}_- \to \mathcal{C}_-$. If $d_1^2 = d_2^2 = 0$, $d^2 = d_1 d_2 + d_2 d_1$. Therefore, d is nilpotent iff the operators d_1, d_2 anti-commute.

Theorem 5. *The operator $d_1 d_2 + d_2 d_1 = 0$ iff the curvature of ∇ vanishes $R = 0$.*

Proof. Applying the definition of d_1 and d_2, one can find that

$$(d_1 d_2 + d_2 d_1)\phi = C_1^\mu \bar{\wedge} C_2^\nu \bar{\wedge} \nabla_\mu \nabla_\nu \phi - C_1^\mu \bar{\wedge} C_2^\nu \bar{\wedge} \nabla_\nu \nabla_\mu \phi = C_2^\mu \bar{\wedge} C_1^\nu \bar{\wedge} R_{\mu\nu}\phi. \tag{4.10}$$

The Kähler form associated with the parallel spinors is

$$\Omega = \frac{-i}{2C(\zeta_2, \zeta_1)} C(\zeta_2, \Gamma_{\mu\nu}\zeta_1) e^\mu \wedge e^\nu. \tag{4.11}$$

It can then be seen that

$$C\Gamma_{(2p)}(\zeta_2, \zeta_1) = \frac{1}{(2p)!} C(\zeta_2, \Gamma_{\rho_1 \ldots \rho_{2p}}\zeta_1) e^{\rho_1} \wedge \ldots \wedge e^{\rho_{2p}} = \frac{(-i)^p C(\zeta_2, \zeta_1)}{p!} \wedge^p \Omega. \tag{4.12}$$

Applying (4.9), we find

$$\begin{aligned}(d_1 d_2 + d_2 d_1)\phi &= \frac{1}{\dim_{\mathbb{C}}(\Delta_+)} \sum_{p=0}^{\frac{m}{2}} \frac{(-1)^{s_C + s_\Gamma}}{(2p)!} C(\zeta_2, \Gamma^\mu \Gamma^{\rho_1 \cdots \rho_{2p}} \Gamma^\nu \zeta_1) C\Gamma_{\rho_1 \ldots \rho_{2p}} \bar{\wedge} R_{\mu\nu}\phi \\ &= \frac{1}{\dim_{\mathbb{C}}(\Delta_+)} \sum_{p=0}^{\frac{m}{2}} \frac{(-1)^{s_C + s_\Gamma}}{(2p)!} [C(\zeta_2, \Gamma^{\mu\rho_1 \cdots \rho_{2p}\nu}\zeta_1) \\ &\quad + (2p)(2p-1) g^{\mu\rho_1} C(\zeta_2, \Gamma^{\rho_2 \cdots \rho_{2p-1}}\zeta_1) g^{\rho_{2p}\nu}] C\Gamma_{\rho_1 \ldots \rho_{2p}} \bar{\wedge} R_{\mu\nu}\phi,\end{aligned} \tag{4.13}$$

where g is the metric on the manifold M. This can be rewritten as

$$\begin{aligned}(d_1 d_2 + d_2 d_1)\phi &= \frac{C(\zeta_2, \zeta_1)}{\dim_{\mathbb{C}}(\Delta_+)} \sum_{p=0}^{\frac{m}{2}} \frac{(-1)^{s_C + s_\Gamma}}{(2p)!} \left[\frac{(-i)^{p+1}(2p+2)!}{2^{p+1}(2p+1)(p+1)!} \right. \\ &\quad \times \Omega^{\mu\nu} \Omega^{\rho_1\rho_2} \ldots \Omega^{\rho_{2p-1}\rho_{2p}} C\Gamma_{\rho_1 \ldots \rho_{2p}} \bar{\wedge} R_{\mu\nu}\phi + \frac{(-i)^{p-1}(2p)!}{2^{p-1}(p-1)!} \\ &\quad \left. \times \Omega^{\rho_2\rho_3} \ldots \Omega^{\rho_{2p-2}\rho_{2p-1}} C\Gamma_{\rho_1 \ldots \rho_{2p}} \bar{\wedge} (R_{\mu\nu} J^\mu_{\rho_1} J^\nu_{\rho_{2p}} + R_{\rho_1\rho_{2p}})\phi \right]\end{aligned} \tag{4.14}$$

For $m = 4k$, $C\Gamma_{(2p)}$ is symmetric for $p = 2q$ while they are skew-symmetric for $p = 2q + 1$. Therefore, in this case only the latter terms contribute in the sum. Similarly for $m = 4k + 2$, $C\Gamma_{(2p)}$ is skew-symmetric for $p = 2q$ while they are symmetric for $p = 2q + 1$. Therefore, only the former terms contribute is the sum.

It is clear that if the (1,1) part of the curvature vanishes, then the proposition is satisfied. However, the (2,0) and the (0,2) parts of the curvature vanish as well. Thus, $R = 0$. □

We therefore conclude that there are three kinds of untwisted differential spin cohomology associated with a manifold that admits a connection with holonomy contained in $SU(m)$, $m = 4k, 4k + 2$. The complexes are $(\mathcal{C}_-, d_1)$, $(\mathcal{C}_-, d_2)$ and $(\mathcal{C}_-, d = d_1 + d_2)$ and the associated spin cohomologies are denoted as $H_{d_1}(\mathcal{C}_-)$, $H_{d_2}(\mathcal{C}_-)$ and $H_d(\mathcal{C}_-)$, respectively. Unlike the case case where $m = 4k+1, 4k+3$, all three cohomologies are cohomologies of the complex $\mathcal{C}_-$.

4.2. Adjoint operators and Laplacians

As we have mentioned in Section 2, $\Delta_\pm$ are equipped with a $Spin(n)$-invariant inner product. Because of this, one can find the adjoints of the spin cohomology operators d_1, d_2 and their associated Laplacians. As in the previous section, we shall distinguish between the $m = 4k+1$, $4k+3$ and $m = 4k$, $4k+2$ cases. This is because of the properties of the inner product are different (see Section 2).

4.3. The $m = 4k+1$, $4k+3$ case

We extend the inner product C^{-1} from $\Delta_+ \oplus \Delta_-$ to the space of sections of $\mathcal{C}_+ \oplus \mathcal{C}_-$ and denote it with the same symbol. The inner product C^{-1} vanishes if it is restricted on either $\mathcal{C}_+$ or $\mathcal{C}_-$.

Definition 6. The adjoint operator $\Delta_1 : \mathcal{C}_- \to \mathcal{C}_-$ of $d_1 : \mathcal{C}_+ \to \mathcal{C}_+$ is

$$C^{-1}(\phi, d_1\psi) = C^{-1}(\delta_1\phi, \psi). \tag{4.15}$$

Similarly, the adjoint operator $\Delta_2 : \mathcal{C}_+ \to \mathcal{C}_+$ of $d_2 : \mathcal{C}_- \to \mathcal{C}_-$ is

$$C^{-1}(\psi, d_2\phi) = C^{-1}(\delta_2\psi, \phi). \tag{4.16}$$

Using these adjoints, one can define two Laplace operators $\Delta_1 = \Delta_2 d_1 + d_1\Delta_2$ and $\Delta_2 = \Delta_1 d_2 + d_2\Delta_1$. The Laplace operator of $d : \mathcal{C}_+ \oplus \mathcal{C}_- \to \mathcal{C}_+ \oplus \mathcal{C}_-$ is $\Delta = \Delta_1 \oplus \Delta_2$.

To compute the Laplace operator Δ_1, we use the above definitions to find

$$\Delta_1\phi = -(-1)^{s_C+s_\Gamma}\left(C(\zeta_1, \Gamma^\nu\Gamma^\mu\zeta_2)\nabla_\mu\nabla_\nu\phi + (-1)^{s_c+s_\Gamma}C_1^\nu\bar{\wedge}(C_2^\mu\lrcorner R_{\nu\mu}\phi)\right), \tag{4.17}$$

where $\eta\lrcorner\phi$ denotes inner derivation with respect to spinor η, i.e.

$$\eta\lrcorner\phi = \frac{(-1)^{s_c}}{\ell!}\eta^B\phi_{BA_1\dots A_\ell}\epsilon^{A_1}\bar{\wedge}\dots\bar{\wedge}\epsilon^{A_\ell} = \frac{(-1)^{s_c}}{\ell!}(C^{-1})^{BE}\eta_B\phi_{EA_1\dots A_\ell}\epsilon^{A_1}\bar{\wedge}\dots\bar{\wedge}\epsilon^{A_\ell}, \tag{4.18}$$

which is equivalent to

$$(\eta\lrcorner\phi)_{A_1\dots A_\ell} = (\ell+1)(-1)^{s_c}\eta^B\phi_{BA_1\dots A_\ell}. \tag{4.19}$$

More generally, we have

$$(\eta\lrcorner\phi)_{E_1\dots E_q,A_1\dots A_\ell} = (\ell+1)(-1)^{s_c}\eta^B_{E_1\dots E_q}\phi_{BA_1\dots A_\ell} \tag{4.20}$$

The product of the co-spinor representations $\Delta_\pm$ can be decomposed as

$$\Delta_+ \otimes \Delta_- = \sum_{p=0}^{[m/2]} \Lambda^{2p}. \tag{4.21}$$

The formula that relates the product of two co-spinors to forms is given by (2.12) after the appropriate restrictions. Applying this to the second term in the Laplace operator, we find

$$\Delta_1\phi = -(-1)^{s_C+s_\Gamma}\left(C(\zeta_1,\Gamma^\nu\Gamma^\mu\zeta_2)\nabla_\mu\nabla_\nu\phi + \frac{1}{\dim_{\mathbb{C}}(\Delta_+)}\sum_{p=0}^{[\frac{m}{2}]}\frac{1}{(2p)!}\right.$$
$$\left.\times\, C(\zeta_1,\Gamma^\mu\Gamma^{\rho_1\dots\rho_{2p}}\Gamma^\nu\zeta_2)C\Gamma_{\rho_1\dots\rho_{2p}}\lrcorner R_{\mu\nu}\phi\right). \tag{4.22}$$

In turn this can be written as

$$\Delta_1\phi = -(-1)^{s_C+s_\Gamma}C(\zeta_1,\zeta_2)\left((g^{\mu\nu}+i\Omega^{\mu\nu})\nabla_\mu\nabla_\nu\phi + \frac{1}{\dim_{\mathbb{C}}(\Delta_+)}\sum_{p=0}^{[\frac{m}{2}]}\frac{1}{(2p)!}\right.$$
$$\times\left[\frac{i^{p+1}(2p+2)!}{2^{p+1}(2p+1)(p+1)!}\Omega^{\mu\nu}\Omega^{\rho_1\rho_2}\dots\Omega^{\rho_{2p-1}\rho_{2p}}C\Gamma_{\rho_1\dots\rho_{2p}}\lrcorner R_{\mu\nu}\phi + \frac{i^{p-1}(2p)!}{2^{p-2}(p-1)!}\right.$$
$$\left.\left.\times\,\Omega^{\rho_2\rho_3}\dots\Omega^{\rho_{2p-2}\rho_{2p-1}}g^{\rho_1\mu}g^{\rho_{2p}\nu}C\Gamma_{\rho_1\dots\rho_{2p}}\lrcorner R^{1,1}_{\mu\nu}\phi\right]\right), \tag{4.23}$$

where

$$R^{1,1}_{\rho_1\rho_{2p}} = \frac{1}{2}(R_{\mu\nu}J^\mu_{\rho_1}J^\nu_{\rho_{2p}} + R_{\rho_1\rho_{2p}}). \tag{4.24}$$

is the (1,1) part of the curvature with respect to the almost complex structure J.

The Laplace operators Δ_2 is given as in (4.17) but with the parallel spinors ζ_1 and ζ_2 interchanged. The effect that this can be easily computed from 4.17) using the symmetry properties of $C\Gamma_{\mu_1\dots\mu_q}$. In particular, we find that

Corollary 2. *The Δ_2 Laplace operator is*

$$\Delta_2\phi = -(-1)^{s_C+s_\Gamma}C(\zeta_1,\zeta_2)\left((-1)^{s_c}(g^{\mu\nu}-i\Omega^{\mu\nu})\nabla_\mu\nabla_\nu\phi + \frac{(-1)^{s_c}}{\dim_{\mathbb{C}}(\Delta_+)}\right.$$
$$\times\sum_{p=0}^{[\frac{m}{2}]}\left[\frac{(-i)^{p+1}(2p+2)!}{2^{p+1}(2p+1)(p+1)!}\Omega^{\mu\nu}\Omega^{\rho_1\rho_2}\dots\Omega^{\rho_{2p-1}\rho_{2p}}C\Gamma_{\rho_1\dots\rho_{2p}}\lrcorner R_{\mu\nu}\phi\right.$$
$$\left.\left.+\,\frac{(-i)^{p-1}(2p)!}{2^{p-2}(p-1)!}\Omega^{\rho_2\rho_3}\dots\Omega^{\rho_{2p-2}\rho_{2p-1}}g^{\rho_1\mu}g^{\rho_{2p}\nu}C\Gamma_{\rho_1\dots\rho_{2p}}\lrcorner R^{1,1}_{\mu\nu}\phi\right]\right). \tag{4.25}$$

4.4. The $m = 4k$, $4k+2$ case

In this case, $d_1, d_2 : \mathcal{C}_- \to \mathcal{C}_-$ and the inner product C^{-1} when restricted on $\mathcal{C}_-$ is non-degenerate. We define the adjoints $\delta_1, \delta_2 : \mathcal{C}_- \to \mathcal{C}_-$ of the d_1, d_2 operators, respectively. Again there are two Laplace operators $\Delta_1 = \delta_2 d_1 + d_1\delta_2$ and $\Delta_2 = \delta_1 d_2 + d_2\delta_1$ The expressions of these operators are the same as those in the previous section. In particular, we have that

Corollary 3. *The Δ_1 Laplace operator is*

$$\Delta_1\phi = -(-1)^{s_C+s_\Gamma} C(\zeta_1,\zeta_2)\Bigg((g^{\mu\nu}+i\Omega^{\mu\nu})\nabla_\mu\nabla_\nu\phi + \frac{1}{\dim_{\mathbb{C}}(\Delta_+)}\sum_{p=0}^{\frac{m}{2}}\frac{1}{(2p)!} \times\left[\frac{i^{p+1}(2p+2)!}{2^{p+1}(2p+1)(p+1)!}\Omega^{\mu\nu}\Omega^{\rho_1\rho_2}\ldots\Omega^{\rho_{2p-1}\rho_{2p}}C\Gamma_{\rho_1\ldots\rho_{2p}}\lrcorner R_{\mu\nu}\phi + \frac{i^{p-1}(2p)!}{2^{p-2}(p-1)!}\Omega^{\rho_2\rho_3}\ldots\Omega^{\rho_{2p-2}\rho_{2p-1}}g^{\rho_1\mu}g^{\rho_{2p}\nu}C\Gamma_{\rho_1\ldots\rho_{2p}}\lrcorner R^{1,1}_{\mu\nu}\phi\right]\Bigg). \quad (4.26)$$

Similarly, the Δ_2 Laplace operator is

$$\Delta_2\phi = -(-1)^{s_C+s_\Gamma} C(\zeta_1,\zeta_2)\Bigg((-1)^{s_C}(g^{\mu\nu}-i\Omega^{\mu\nu})\nabla_\mu\nabla_\nu\phi + \frac{(-1)^{s_C}}{\dim_{\mathbb{C}}(\Delta_+)} \times\sum_{p=0}^{\frac{m}{2}}\left[\frac{(-i)^{p+1}(2p+2)!}{2^{p+1}(2p+1)(p+1)!}\Omega^{\mu\nu}\Omega^{\rho_1\rho_2}\ldots\Omega^{\rho_{2p-1}\rho_{2p}}C\Gamma_{\rho_1\ldots\rho_{2p}}\lrcorner R_{\mu\nu}\phi + \frac{(-i)^{p-1}(2p)!}{2^{p-2}(p-1)!}\Omega^{\rho_2\rho_3}\ldots\Omega^{\rho_{2p-2}\rho_{2p-1}}g^{\rho_1\mu}g^{\rho_{2p}\nu}C\Gamma_{\rho_1\ldots\rho_{2p}}\lrcorner R^{1,1}_{\mu\nu}\phi\right]\Bigg). \quad (4.27)$$

One could also define two more Laplace operators $\hat{\Delta}_1 = \delta_1 d_1 + d_1\delta_1$ and $\hat{\Delta}_2 = \delta_2 d_2 + d_2\delta_2$. However, they vanish. This can be seen by a direct computation

$$\hat{\Delta}_1\phi = -(-1)^{s_C+s_\Gamma}\Bigg(C(\zeta_1,\Gamma^\nu\Gamma^\mu\zeta_1)\nabla_\mu\nabla_\nu\phi + \frac{1}{\dim_{\mathbb{C}}(\Delta_+)}\sum_{p=0}^{\frac{m}{2}}\frac{1}{(2p)!} \times\left[C(\zeta_1,\Gamma^{\mu\rho_1\ldots\rho_{2p}\nu}\zeta_1)+(2p)(2p-1)g^{\mu\rho_1}C(\zeta_1,\Gamma^{\rho_2\ldots\rho_{2p-1}}\zeta_1)g^{\rho_{2p}\nu}\right] \times C\Gamma_{\rho_1\ldots\rho_{2p}}\lrcorner R_{\mu\nu}\phi = 0\Bigg), \quad (4.28)$$

because the (0,2) part of the curvature R vanishes. Similarly $\hat{\Delta}_2 = 0$.

5. *SU*(*m*) holonomy, twisted complexes and algebraic spin cohomology

5.1. Twisted $\mathcal{C}_\pm \otimes E$.

Let M a Riemannian manifold equipped with a connection ∇^M such that $\mathrm{hol}(\nabla^M) \subseteq SU(m)$. In addition, let E be a vector bundle over M equipped with a connection ∇^E and associated curvature F.

As in the previous section one can construct two first order differential spin operators d_1 and d_2 associated with the two parallel spinors of ∇^M. Then one can use the vector bundle E to twist the complexes $\mathcal{C}_+$ and $\mathcal{C}_-$ as $\mathcal{C}_+\otimes E$ and $\mathcal{C}_-\otimes E$, respectively. Furthermore, one can use the connection ∇^E to extend d_1 and d_2 to the twisted complexes as it has been described in (3.9). We denote the extended operators with the same symbols. In particular, we have

Corollary 4. *The operators* $d_1 : \mathcal{C}_+ \otimes E \to \mathcal{C}_+ \otimes E$ *for* $m = 4k+1, 4k+3$ *and* $d_1 : \mathcal{C}_- \otimes E \to \mathcal{C}_- \otimes E$ *for* $m = 4k, 4k+2$ *are nilpotent*, $d_1^2 = 0$, if the (2,0) *part of the curvature*, R *and* F, *of both the connections* ∇ *and* ∇^E *vanishes. Similarly, The operator* $d_2 : \mathcal{C}_- \otimes E \to \mathcal{C}_- \otimes E$, $m = 4k, 4k+1, 4k+2, 4k+3$, *is nilpotent, if the* (0,2) *part of the curvature*, R *and* F, *of both the connections* ∇ *and* ∇^E *vanishes. For* $m = 4k, 4k+2$, $d_1 d_2 + d_2 d_1 = 0$, *if* $F = R = 0$.

The Laplace operators can be easily computed. In particular, we find that

$$\begin{aligned}\Delta_1\phi = -(-1)^{s_C+s_\Gamma} C(\zeta_1,\zeta_2)\Bigg((g^{\mu\nu} + i\Omega^{\mu\nu})\nabla_\mu\nabla_\nu\phi + \frac{1}{\dim_{\mathbb{C}}(\Delta_+)}\sum_{p=0}^{[\frac{m}{2}]}\frac{1}{(2p)!}\\ \times\Bigg[\frac{i^{p+1}(2p+2)!}{2^{p+1}(2p+1)(p+1)!}\Omega^{\mu\nu}\Omega^{\rho_1\rho_2}\ldots\Omega^{\rho_{2p-1}\rho_{2p}}C\Gamma_{\rho_1\ldots\rho_{2p}}\daleth(R_{\mu\nu}+F_{\mu\nu})\phi\\ + \frac{i^{p-1}(2p)!}{2^{p-2}(p-1)!}\Omega^{\rho_2\rho_3}\ldots\Omega^{\rho_{2p-2}\rho_{2p-1}}g^{\rho_1\mu}g^{\rho_{2p}\nu}C\Gamma_{\rho_1\ldots\rho_{2p}}\daleth(R^{1,1}_{\mu\nu}+F^{1,1}_{\mu\nu})\phi\Bigg]\Bigg).\end{aligned} \tag{5.1}$$

Similarly, the Δ_2 Laplace operator is

$$\begin{aligned}\Delta_2\phi = -(-1)^{s_C+s_\Gamma} C(\zeta_1,\zeta_2)\Bigg((-1)^{s_c}(g^{\mu\nu} - i\Omega^{\mu\nu})\nabla_\mu\nabla_\nu\phi + \frac{(-1)^{s_c}}{\dim_{\mathbb{C}}(\Delta_+)}\\ \times\sum_{p=0}^{[\frac{m}{2}]}\Bigg[\frac{(-i)^{p+1}(2p+2)!}{2^{p+1}(2p+1)(p+1)!}\Omega^{\mu\nu}\Omega^{\rho_1\rho_2}\ldots\Omega^{\rho_{2p-1}\rho_{2p}}C\Gamma_{\rho_1\ldots\rho_{2p}}\daleth(R_{\mu\nu}+F_{\mu\nu})\phi\\ + \frac{(-i)^{p-1}(2p)!}{2^{p-2}(p-1)!}\Omega^{\rho_2\rho_3}\ldots\Omega^{\rho_{2p-2}\rho_{2p-1}}g^{\rho_1\mu}g^{\rho_{2p}\nu}C\Gamma_{\rho_1\ldots\rho_{2p}}\daleth(R^{1,1}_{\mu\nu}+F^{1,1}_{\mu\nu})\phi\Bigg]\Bigg).\end{aligned} \tag{5.2}$$

5.2. *Twisted* $\mathcal{C}_\pm(E)$ *complexes*

Suppose that the parallel spinors with respect to the $\mathcal{D}$ connection on $\Delta_\pm \otimes E$ are in the direction of either $1 \otimes 1$ or $e^1 \wedge \ldots \wedge e^m \otimes 1$ and E is a vector bundle with a fibre inner product h, $\nabla^E h = 0$. One can construct an invariant inner product on $\Delta_\pm \otimes E$ as $C^{-1} \otimes h$ and extended to the twisted complexes $\mathcal{C}_\pm(E)$. We can again define operators d_1 and d_2. In particular, we have

Proposition 9. *The operators* $d_1 : \mathcal{C}_+(E) \to \mathcal{C}_+(E)$ *for* $m = 4k+1, 4k+3$ *and* $d_1 : \mathcal{C}_-(E) \to \mathcal{C}_-(E)$ *for* $m = 4k, 4k+2$ *are nilpotent*, $d_1^2 = 0$, *if the* (2,0) *part of the curvature of* $\mathcal{D}$ *vanishes. Similarly, The operator* $d_2 : \mathcal{C}_-(E) \to \mathcal{C}_-(E)$, $m = 4k, 4k+1, 4k+2, 4k+3$, *is nilpotent*, $d_2^2 = 0$, *if the* (0,2) *part of the curvature of* $\mathcal{D}$ *vanishes. For* $m = 4k, 4k+2$, $d_1 d_2 + d_2 d_1 = 0$, *if the curvature of* $\mathcal{D}$ *vanishes.*

Proof. The proof is similar to that we have already investigated in the previous sections for the untwisted d_1 and d_2 operators. However, there is one difference. If E is not a line bundle, then in the expression for d_1^2 and d_2^2 both symmetric and skew-symmetric $C\Gamma^{(p)}$ contribute. This is unlike the untwisted case where only the skew-symmetric $C\Gamma^{(p)}$ contribute. However, there is no additional restriction on the curvature of $\mathcal{D}$. □

The Laplace operators Δ_1 and Δ_2 can be easily computed in this case. The expressions are as in (5.1) and (5.2) with the curvature R and F replaced by the curvature of $\mathcal{D}$.

5.3. Algebraic cohomologies

We have seen that the operator D on the complex $\Lambda^* \otimes \mathcal{C}$ is defined provided that $C\Gamma$ is skew-symmetric which is the case for $C = A$ if $m = 4k, 4k + 3$ and for $C = B$ if $m = 4k + 2$. Moreover, D restricts on $\mathcal{C}_\pm$, $D : \mathcal{C}_\pm \to \mathcal{C}_\pm$ if $m = 4k + 3$. Therefore, we conclude that the D operator can be defined on the complexes $\Lambda^* \otimes \mathcal{C}_\pm$ and $\Lambda^* \otimes \mathcal{C}_\pm \otimes E$ only for $m = 4k + 3$. The operator D can also defined for twisted complexes $\Lambda^* \otimes \mathcal{C}_\pm(E)$ but we shall not investigate this further here.

Corollary 5. *The algebraic operator D anticommutes with both d_1 and d_2 differential operators*

$$d_1 D + D d_1 = d_2 D + D d_2 = 0. \tag{5.3}$$

Therefore, one can define the operators $d_1 + D$ and $d_2 + D$ which are nilpotent provided $d_1^2 = d_2^2 = 0$. The cohomology of $d_1 + D$ and $d_2 + D$ can be computed using spectral sequences. We shall not do a general computation. Instead, we shall give the cohomology of the operator $d_2 + D$ in the special case where M is a six-dimensional Calabi-Yau manifold.

6. Complex manifolds with holonomy $SU(m)$ and spin cohomology

It is clear from the results of the previous section that complex spin cohomology is related to the Dolbeault cohomology. Here, we shall establish the precise relation and we shall give the classes of the spin cohomology in terms of those of the Dolbeault cohomology, see e.g. [11,12].

6.1. Spin and Dolbeault cohomologies

Let M be a complex manifold equipped with a connection ∇, $\mathrm{hol}(\nabla) \subseteq SU(m)$. On M, it is known that

$$\Delta = \oplus_q \Lambda^{0,q} = \Lambda^{0,*}. \tag{6.1}$$

This can be easily seen from $\Delta = \Lambda^{0,q}(1)$, where $\Lambda^{0,q}$ acts on 1 with Clifford multiplication. In particular, we have $\Delta_+ = \Lambda^{0,\mathrm{even}}$ and $\Delta_- = \Lambda^{0,\mathrm{odd}}$. Thus,

$$\begin{aligned} \mathcal{C}_+ &= \Lambda^*(\Lambda^{0,\mathrm{even}}) \\ \mathcal{C}_- &= \Lambda^*(\Lambda^{0,\mathrm{odd}}) \end{aligned} \tag{6.2}$$

Write $\Delta_- = \Lambda^{0,1} \oplus Z$, where $Z = \oplus_{p\geq 1} \Lambda^{0,2p+1}$. The complex $\mathcal{C}_-$ can now be decomposed as

$$\mathcal{C}_-^\ell = \oplus_{p+q=\ell} \Lambda^{0,p} \otimes \Lambda^q(Z). \tag{6.3}$$

Proposition 10.

$$d_2 = \bar\partial : \Lambda^{0,p} \otimes \Lambda^q(Z) \to \Lambda^{0,p+1} \otimes \Lambda^q(Z). \tag{6.4}$$

Proof. To show this, we first evaluate the action of d_2 on $\Lambda^{0,1} \subset \mathcal{C}_-$. Indeed let $\eta_i e^i \in \Lambda^{0,1}$, then we have

$$d_2(\eta_i e^i) = \Lambda(-1)^{\frac{1}{2}m(m-1)}(\nabla_i + i\nabla_{i+m})\eta_j e^i \bar\wedge e^j = \Lambda(-1)^{\frac{1}{2}m(m-1)}\bar\partial_i \eta_j e^i \wedge e^j, \tag{6.5}$$

where $\Lambda = 1$ for the A inner product and $\Lambda = i^m$ for B inner product. After suppressing the numerical coefficient which is inconsequential for the computation of cohomology, we have $d_2 = \bar\partial$ on $\Lambda^{0,1}$. Using the definition of the $\bar\wedge$ product, it is straightforward to extend the proof to the rest of the complex $\mathcal{C}_-$. $\square$

Corollary 6. *Let M be a complex manifold as described in the beginning of the section. Then the spin cohomology*

$$H^\ell_{d_2}(\mathcal{C}_-) = \oplus_{p+q=\ell} H^{0,p}_{\bar\partial}(\Lambda^q(Z)). \tag{6.6}$$

Therefore, the spin cohomology of the d_2 operator can be computed in terms of Dolbeault cohomology of a twisted complex by the bundle $\Lambda^*(Z)$, where $Z = \oplus_{p\geq 1}\Lambda^{0,2p+1}$.

A direct consequence of this is that

$$H^\ell_{d_2}(\mathcal{C}_- \otimes E) = \oplus_{p+q=\ell} H^{0,p}_{\bar\partial}(\Lambda^q(Z) \otimes E). \tag{6.7}$$

Using the corollary, we can also compute the index of the spin complex $(\mathcal{C}_-, d_2)$ in terms of the index of the twisted $\bar\partial$ complex. In particular, we have

$$\mathrm{Index}_{d_2}(\mathcal{C}_-) = \sum_{q\geq 0}(-1)^q \mathrm{Index}_{\bar\partial}(\Lambda^q(Z)) \tag{6.8}$$

or more generally

$$\mathrm{Index}_{d_2}(\mathcal{C}_- \otimes E) = \sum_{q\geq 0}(-1)^q \mathrm{Index}_{\bar\partial}(\Lambda^q(Z) \otimes E). \tag{6.9}$$

It remains to investigate the cohomology of d_1. We shall consider the cases $m = 4k+1, 4k+3$ and $m = 4k, 4k+2$ separately. In the former case $d_1 : \mathcal{C}_+ \to \mathcal{C}_+$. Writing $\Delta_+ = \Lambda^{0,m-1} \oplus W$, where $W = \oplus_{p<\frac{m-1}{2}}\Lambda^{0,2p}$, we have

$$\mathcal{C}^\ell_+ = \oplus_{p+q=\ell}\Lambda^p(\Lambda^{0,m-1}) \otimes \Lambda^q(W), \tag{6.10}$$

and

$$d_1 : \Lambda^p(\Lambda^{0,m-1}) \otimes \Lambda^q(W) \to \Lambda^{p+1}(\Lambda^{0,m-1}) \otimes \Lambda^q(W). \tag{6.11}$$

Since there is a $SU(m)$ structure, we can identify $\Lambda^{0,m-1} = \Lambda^{1,0}$, $\Lambda^p(\Lambda^{0,m-1}) = \Lambda^{p,0}$ and

$$d_1 = \partial : \Lambda^{p,0} \otimes \Lambda^q(W) \to \Lambda^{p+1,0} \otimes \Lambda^q(W). \tag{6.12}$$

Therefore, we conclude that

$$H^\ell_{d_1}(\mathcal{C}_+) = \oplus_{p+q=\ell} H^{p,0}_{\partial}(\Lambda^q(W)). \tag{6.13}$$

Thus, $H^*_{d_1}(\mathcal{C}_+) = H^*_{d_2}(\mathcal{C}_-)$. The same applies for $m = 4k, 4k+2$.

6.2. Algebraic operations on twisted complexes

On complex manifolds with $\mathrm{hol}(\nabla) \subseteq SU(m)$ apart from the differential spin operators d_1, d_2, there is also an algebraic spin operator D provided $m = 4k + 3$. We shall focus on the twisted spin cohomology associated with the operators d_2 and D. In the context of complex geometry, there are several versions that one can consider. In particular, one can defined a twisted spin cohomology on $\Lambda^* \otimes \mathcal{C}_-$ as we have already mentioned in Section 5.3. However, it is also possible to twist $\mathcal{C}_-$ with either $\Lambda^{*,0}$ or $\Lambda^{0,*}$. In each of these cases the twisted spin cohomology of the operator $d_2 + D$, or equivalently $\bar{\partial} + D$, can be computed using a spectral sequence.

The twisted complex $C = \Lambda^* \otimes \mathcal{C}_-$ is a double complex, $C^{p,\ell}$, with grading induced from the space of forms Λ^* and that of $\mathcal{C}_-$. However, in this grading, $d_2 = \bar{\partial}$ and D do not act with horizontal and vertical operations. In particular, $d_2 : C^{p,\ell} \to C^{p,\ell+1}$ and $D : C^{p,\ell} \to C^{q-1,\ell+2}$. It is therefore convenient to introduce a new grading as

$$C^{[-p,\ell+2p]} = C^{p,\ell} = \Lambda^p \otimes \mathcal{C}_-^\ell. \tag{6.14}$$

Note know that $d_1 : C^{[-p,\ell+2p]} \to C^{[-p,\ell+2p+1]}$ and $D : C^{[-p,\ell+2p]} \to C^{[-p+1,\ell+2p]}$ as expected. The twisted complexes $\Lambda^{*,0} \otimes \mathcal{C}_-$ and $\Lambda^{0,*} \otimes \mathcal{C}_-$ can be treated in a similar way. The machinery of spectral sequences can now be used to do the computation, see e.g [10] and references within. Instead of developing the general theory of computing the cohomology of $\bar{\partial} + D$ for the various complexes above, we shall give the cohomology of $(\Lambda^{*,0} \otimes \mathcal{C}_-, \bar{\partial} + D)$ for six-dimensional Calabi-Yau manifolds in an example below.

6.3. $Spin_c$ structures and spin cohomology

Let M be a complex manifold equipped with a *spin$_c$* structure and compatible connection ∇, $\mathrm{hol}(\nabla) \subseteq SU(m)$. Suppose that L is a (locally defined) complex line bundle associated with the *spin$_c$* structure. On M, it is known that

$$\Delta^* \otimes L = \oplus_q \Lambda^{0,q} = \Lambda^{0,*}. \tag{6.15}$$

This is similar to the standard complex case, we have investigated. In particular, we have $\Delta_+ \otimes L = \Lambda^{0,\mathrm{even}}$ and $\Delta_- = \Lambda^{0,\mathrm{odd}}$. Thus,

$$\begin{aligned} \mathcal{C}_+(L) &= \Lambda^*(\Lambda^{0,\mathrm{even}}) \\ \mathcal{C}_-(L) &= \Lambda^*(\Lambda^{0,\mathrm{odd}}) \end{aligned} \tag{6.16}$$

Write $\Delta_- \otimes L = \Lambda^{0,1} \otimes L \oplus Z \otimes L$, where $Z = \oplus_{p\geq 1} \Lambda^{0,2p+1}$. The complex $\mathcal{C}_-(L)$ can now be decomposed as

$$\mathcal{C}_-^\ell(L) = \oplus_{p+q=\ell} \Lambda^p(\Lambda^{0,1}) \otimes \Lambda^q(Z). \tag{6.17}$$

Proposition 11.

$$d_2 = \bar{\partial} : \Lambda^p(\Lambda^{0,1}) \otimes \Lambda^q(Z \otimes L) \to \Lambda^{p+1}(\Lambda^{0,1}) \otimes \Lambda^q(Z). \tag{6.18}$$

Proof. To show this, we first evaluate the action of d_2 on $\Lambda^{0,1} \subset \mathcal{C}_-(L)$. Indeed let $\eta_i e^i \in \Lambda^{0,1}$, then we have

$$d_2(\eta_i e^i) = \Lambda(-1)^{(1/2)m(m-1)}(\nabla_i + i\nabla_{i+m})\eta_j e^i \bar{\wedge} e^j = \Lambda(-1)^{(1/2)m(m-1)}\bar{\partial}_i \eta_j e^i \wedge e^j, \tag{6.19}$$

where $\Lambda = 1$ for the A inner product and $\Lambda = i^m$ for B inner product. After suppressing the numerical coefficient which is inconsequential for the computation of cohomology, we have

$d_2 = \bar{\partial}$ on $\Lambda^{0,1}$. Using the definition of the $\bar{\wedge}$ product, it is straightforward to extend the proof to the rest of the complex $\mathcal{C}_-(L)$. □

Corollary 7. *Let M be a complex $Spin_c$ manifold as described in the beginning of the section. Then the spin cohomology*

$$H^\ell_{d_2}(\mathcal{C}_-(L)) = \oplus_{p+q=\ell} H^{0,p}_{\bar{\partial}}(\Lambda^q(Z)). \tag{6.20}$$

Therefore, the spin cohomology of the d_2 operator can be computed in terms of Dolbeault cohomology of a twisted complex by the bundle $L^p \otimes \Lambda^q(Z)$, where $Z = \oplus_{p\geq 1}\Lambda^{0,2p+1}$.

7. Spin cohomology and six-dimensional Calabi-Yau manifolds

7.1. Differential spin cohomology

Applying the general theory of the previous section to this case, we have $\Delta_- = \Lambda^{0,\mathrm{odd}} = \Lambda^{0,1} \oplus \Lambda^{0,3}$. In addition for six-dimensional Calabi-Yau manifolds $\Lambda^{0,3}$ is trivial line bundle. Using these, we find that

$$\mathcal{C}^\ell_- = \Lambda^{0,\ell} \oplus \Lambda^{0,\ell-1} \tag{7.1}$$

and

$$d_2 = \bar{\partial} : \Lambda^{0,\ell} \oplus \Lambda^{0,\ell-1} \to \Lambda^{0,\ell+1} \oplus \Lambda^{0,\ell}. \tag{7.2}$$

Therefore,

$$H^\ell_{d_2}(\mathcal{C}_-) = H^{0,\ell}_{\bar{\partial}} \oplus H^{0,\ell-1}_{\bar{\partial}}. \tag{7.3}$$

In particular for an irreducible six-dimensional Calabi-Yau manifold, we have that

$$H^0_{d_2} = \mathbb{C}, \quad H^1_{d_2} = \mathbb{C}, \quad H^2_{d_2} = 0, \quad H^3_{d_2} = \mathbb{C}, \quad H^4_{d_2} = \mathbb{C}. \tag{7.4}$$

It is also straightforward to compute the cohomology of the twisted complex $(\Lambda^{*,0} \otimes \mathcal{C}_-, d_2)$. In particular, we find that

$$H^{p,\ell}_{d_2}(\Lambda^{*,0} \otimes \mathcal{C}_-) = H^{p,\ell}_{\bar{\partial}} \oplus H^{p,\ell-1}_{\bar{\partial}}. \tag{7.5}$$

7.2. Twisted complexes and algebraic spin cohomology

To compute the cohomology of the complex $(\Lambda^{*,0} \otimes \mathcal{C}_-, d_2 + D)$, we first investigate the complex $(\Lambda^{*,0} \otimes \mathcal{C}_-, D)$. Using (7.1), we find that the operator $D : \Lambda^{p,0} \otimes \mathcal{C}^\ell_- \to \Lambda^{p-1,0} \otimes \mathcal{C}^{\ell+2}_-$ acts as

$$D : \Lambda^{p,0} \otimes [\Lambda^{0,\ell} \oplus \Lambda^{0,\ell-1}] \to \Lambda^{p-1} \otimes [\Lambda^{0,\ell+2} \oplus \Lambda^{0,\ell+1}] \tag{7.6}$$

or equivalently $D : \Lambda^{p,\ell} \oplus \Lambda^{p,\ell-1} \to \Lambda^{p-1,\ell+2} \oplus \Lambda^{p-1,\ell+1}$. Since it acts on the two parts in the sum separately, it is enough to consider only its action in the first part. After some computation, one finds that

$$D\psi = -(-1)^q \frac{1}{(p-1)!q!2} \psi_{\gamma\alpha_1\dots\alpha_{p-1},\bar{\beta}_3\dots\bar{\beta}_{q+2}} \times \epsilon^\gamma_{\bar{\beta}_1\bar{\beta}_2} e^{\alpha_1} \wedge \dots \wedge e^{\alpha_{p-1}} \wedge e^{\bar{\beta}_1} \wedge e^{\bar{\beta}_2} \wedge \dots \wedge e^{\bar{\beta}_{q+2}}, \tag{7.7}$$

where $\psi \in \Lambda^{p,q}$.

Next we shall compute the cohomology of the double complex $(\Lambda^{*,*}, \bar\partial + D)$ using a spectral sequence, see e.g. [10] and references within. The most convenient filtration is that for which $E_1 = H^{p,q}_{\bar\partial}$. Then from the general theory of spectral sequences for double complexes $E_2 = H_D H_{\bar\partial}$ and E_2 is graded as the double complex. It is known that for six-dimensional irreducible Calabi-Yau manifolds the non-vanishing Dolbeault groups are $H^{0,0}_{\bar\partial}$, $H^{1,1}_{\bar\partial}$, $H^{2,1}_{\bar\partial}$, $H^{1,2}_{\bar\partial}$ $H^{2,2}_{\bar\partial}$ and $H^{3,3}_{\bar\partial}$. Moreover, $H^{3,0}_{\bar\partial} = H^{0,3}_{\bar\partial} = \mathbb{C}$ which are generated by the parallel (3,0)- and (0,3)-forms, respectively. To compute E_2 observe that

$$0 \to H^{3,0}_{\bar\partial} \stackrel{D}{\to} H^{2,2}_{\bar\partial} \to 0 \tag{7.8}$$

and

$$0 \to H^{1,1}_{\bar\partial} \stackrel{D}{\to} H^{0,3}_{\bar\partial} \to 0, \tag{7.9}$$

and the rest of the cohomology groups of $E_1 = H^{p,q}_{\bar\partial}$ live in E_2. It is easy to see that $\mathrm{Ker}D|_{H^{3,0}_{\bar\partial}} = \{0\}$. Therefore, $E_2^{3,0}$ vanishes. In addition $D(H^{3,0}_{\bar\partial}) = \mathbb{C}\langle \Omega \wedge \Omega\rangle$, where Ω is the Kähler form. Since $DH^{2,2}_{\bar\partial} = 0$, we conclude that $E_2^{2,2} = H^{2,2}_{\bar\partial}/\mathbb{C}\langle \Omega \wedge \Omega\rangle$, i.e. $E_2^{2,2} = PH^{2,2}_{\bar\partial}$ is generated by the primitive (2,2) harmonic forms.

Next observe that

$$\mathrm{Ker}D|_{H^{1,1}_{\bar\partial}} = \{\alpha \in H^{1,1}_{\bar\partial} \quad \text{such that} \quad \Omega \cdot \alpha = 0\}. \tag{7.10}$$

Therefore, $E_2^{1,1} = PH^{1,1}_{\bar\partial}$ is generated by the primitive (1,1) harmonic forms. In addition we have that $DH^{1,1}_{\bar\partial} = H^{0,3}_{\bar\partial}$, therefore $E_2^{0,3} = 0$. Thus, the only non-vanishing groups are $E_2^{0,0} = \mathbb{C}$, $E_2^{2,1} = H^{2,1}_{\bar\partial}$, $E_2^{1,2} = H^{1,2}_{\bar\partial}$, $E_2^{1,1} = PH^{1,1}_{\bar\partial}$, $E_2^{2,2} = PH^{2,2}_{\bar\partial}$ and $E_2^{3,3} = \mathbb{C}$.

It remains to show that $E_2 = E_\infty$. This is easily verified by computing the action of the differential d_2 of E_2. For this, we need to convert to the grading of the double complex $\Lambda^{*,*}$ for which $\bar\partial : \Lambda^{[m,n]} \to \Lambda^{[m,n+1]}$ acts vertically and $D : \Lambda^{[m,n]} \to \Lambda^{[m+1,n]}$ acts horizontally. As we have explained $(m, n) = (-p, 2p + q)$, i.e. $E_2^{[-p,2p+q]} = H^p_D H^q_{\bar\partial}$. The differential $d_2 : E_2^{[m,n]} \to E_2^{[m-2,n-1]}$. It is easy then to see that the d_2 differential is the zero map and $E_2 = E_\infty$. Therefore, the cohomology of the operator $d + D$ is given by E_2. Thus, we have shown the proposition.

Proposition 12. *Let M be an irreducible six-dimensional Calabi-Yau manifold. The non-vanishing cohomology groups of the complex $(\Lambda^{p,q}, \bar\partial + D)$ are $H^0_{\bar\partial+D} = H^{0,0}_{\bar\partial} = \mathbb{C}$, $H^2_{\bar\partial+D} = PH^{1,1}_{\bar\partial}$, $H^3_{\bar\partial+D} = H^{2,1}_{\bar\partial} \oplus H^{1,2}_{\bar\partial}$, $H^4_{\bar\partial+D} = PH^{2,2}_{\bar\partial}$ and $H^6_{\bar\partial+D} = H^{3,3}_{\bar\partial} = \mathbb{C}$.*

Finally, we have $H^\ell_{\bar\partial+D}(\Lambda^{*,0} \otimes \mathcal{C}_-) = H^\ell_{\bar\partial+D} \oplus H^{\ell-1}_{\bar\partial+D}$.

8. Manifolds with connections with holonomy *Sp*(*k*) and spin cohomology

Let M be a Riemannian manifold which admits a connection ∇ with holonomy $Sp(k)$. In this case, there are $k + 1$ parallel spinors and so $k + 1$ spin differential operators that one can define. The spin differential operators associated with the parallel spinors 1 and $e_1 \wedge \ldots \wedge e_{2k}$ are the same as the d_1 and d_2 spin differential operators that we have investigated for complex manifolds. There are another $k - 1$ spin differential operators d_{Ω^k} associated with the Ω^k, $1 \le k \le k - 1$, parallel spinors of section two. Since $\Omega \in \Lambda^2$ and $m = 2k$ even, $d_{\Omega^k} : \mathcal{C}_- \to \mathcal{C}_-$. We shall not present a complete analysis of all cases. Instead we shall focus on the spin differential operator $d_0 = d_{\Omega^{k-1}}$ associated with the parallel spinor Ω^{k-1}.

As we have explained for complex manifolds, $\mathcal{C}_-$ can be decomposed as

$$\mathcal{C}_-^\ell = \oplus_{p+q=\ell} \Lambda^{0,p} \otimes \Lambda^q(Z). \tag{8.1}$$

Proposition 13. *Let M be a hyperKähler manifold, then*

$$d_0 = K \lrcorner \partial : \Lambda^{0,p} \otimes \Lambda^q(Z) \to \Lambda^{0,p+1} \otimes \Lambda^q(Z), \tag{8.2}$$

where K is the second complex structure on M.

Proof. To show this, we first evaluate the action of d_0 on $\Lambda^{0,1} \subset \mathcal{C}_-$. Indeed let $\eta_i e^i \in \Lambda^{0,1}$, then we have

$$d_0(\eta_i e^i) = \Lambda \langle \omega, \Gamma^i e_k \rangle (\nabla_i - i\nabla_{i+m}) \eta_j e^k \bar{\wedge} e^j = \Lambda K^i_k \partial_i \eta_j e^k \wedge e^j, \tag{8.3}$$

where Λ is inconsequential numerical coefficient that depends on the inner product and the normalization of the parallel spinor and $\langle v, Kw \rangle = \Omega(v, w)$. Thus, we have that $d_2 = K \lrcorner \partial$ on $\Lambda^{0,1}$. Using the definition of the $\bar{\wedge}$ product, it is straightforward to extend the proof to the rest of the complex $\mathcal{C}_-$. □

9. Real spin cohomologies

So far we have investigated the complex spinor cohomologies. Now we shall turn to investigate the real ones. The real spinor representations for $m = 4k, 4k+1, 4k+3$ can be constructed by imposing a reality condition on the complex representations. These reality conditions are

$$\begin{aligned} &\eta = \pm A(\bar{\eta}),\ \eta \in \Delta^\pm, && m = 4k \\ &\eta = A(\bar{\eta}), \quad \eta \in \Delta^+ \oplus \Delta^-, && m = 4k+1 \\ &\eta = B(\bar{\eta}), \quad \eta \in \Delta^+ \oplus \Delta^-, && m = 4k+3 \end{aligned} \tag{9.1}$$

and $\Delta^\pm_{\mathbb{R}} = \{\eta \in \Delta^\pm,\ \text{s.t.}\ \eta = \pm A(\bar{\eta})\}$, $\Delta_{\mathbb{R}} = \{\eta \in \Delta^+ \oplus \Delta^-,\ \text{s.t.}\ \eta = A(\bar{\eta})\}$ and $\Delta_{\mathbb{R}} = \{\eta \in \Delta^+ \oplus \Delta^-,\ \text{s.t.}\ \eta = B(\bar{\eta})\}$, respectively.

9.1. $SU(m)$ invariant spinors

We begin with a summary of the properties of real spin representation in various cases.

9.1.1. m=4k

The real parallel spinors in the $SU(4k)$ case are

$$\tau_1 = \frac{1}{\sqrt{2}}(1 + e_1 \wedge \ldots \wedge e_m), \qquad \tau_2 = \frac{i}{\sqrt{2}}(1 - e_1 \wedge \ldots \wedge e_m). \tag{9.2}$$

Both parallel spinors $\tau_1, \tau_2 \in \Delta^+_{\mathbb{R}}$. We can again define spin cohomologies d_1, d_2 associated with τ_1, τ_2 for $C = A$ on the real complex $\mathcal{C}^{\mathbb{R}}_-$. Thus, $s_c = 0, s_\Gamma = 1$. The decomposition of the real spinor representations is

$$\Delta^\pm_{\mathbb{R}} \otimes \Delta^\pm_{\mathbb{R}} = \sum_{p=0}^{2k-1} \Lambda^{2p}_{\mathbb{R}} \oplus \Lambda^{m\pm}_{\mathbb{R}}. \tag{9.3}$$

9.2. $m = 4k + 1$

The real parallel spinors are

$$\tau_1 = \frac{1}{\sqrt{2}}(1 + e_1 \wedge \ldots \wedge e_m), \quad \tau_2 = \frac{i}{\sqrt{2}}(1 - e_1 \wedge \ldots \wedge e_m). \tag{9.4}$$

The parallel spinors $\tau_1, \tau_2 \in \Delta_{\mathbb{R}}$. We can again define spin cohomologies d_1, d_2 associated with τ_1, τ_2 for $C = A$ on the real complex $\mathcal{C}^{\mathbb{R}}$). Thus, $s_c = 0, s_\Gamma = 0$. The decomposition of the real spinor representations is

$$\Delta_{\mathbb{R}} \otimes \Delta_{\mathbb{R}} = \sum_{p=0}^{2m} \Lambda_{\mathbb{R}}^{2p}. \tag{9.5}$$

9.3. $m = 4k + 3$

The real parallel spinors are

$$\tau_1 = \frac{1}{\sqrt{2}}(1 + ie_1 \wedge \ldots \wedge e_m), \quad \tau_2 = \frac{1}{\sqrt{2}}(i1 + e_1 \wedge \ldots \wedge e_m). \tag{9.6}$$

The parallel spinors $\tau_1, \tau_2 \in \Delta_{\mathbb{R}}$. We can again define spin cohomologies d_1, d_2 associated with τ_1, τ_2 for $C = B$ on the real complex $\mathcal{C}^{\mathbb{R}}$. Thus, $s_c = 0, s_\Gamma = 1$. The decomposition of the real spinor representations is

$$\Delta_{\mathbb{R}} \otimes \Delta_{\mathbb{R}} = \sum_{p=0}^{2m} \Lambda_{\mathbb{R}}^{2p}. \tag{9.7}$$

In all the above cases we find the following:

Theorem 6. *The operators d_1, d_2 are nilpotent iff the curvature of the connection ∇ vanishes.*

Proof. This is a consequence of the results we have already demonstrated in Sections 4 and 6. □

We also have that

Theorem 7. *The Laplacians Δ_1, Δ_2 of the operators d_1, d_2 are*

$$\Delta_2\phi = \Delta_1\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi \tag{9.8}$$

Proof. This is a consequence of the results we have already demonstrated in Section 4. □

Corollary 8. *The real spin cohomologies $H^*(\mathcal{C}^{\mathbb{R}})$ defined above are generated by the parallel elements in $\mathcal{C}^{\mathbb{R}}$ with respect to ∇.*

Proof. This follows from a partial integration formula and the fact that the inner product $C = A, B$ restricted in $\Delta_{\mathbb{R}}$ is definite. □

A class of manifolds which we can define a real spin cohomology are group manifolds equipped with the left or right invariant connections. One can also defined twisted real spin cohomology but we shall not pursue this further here.

10. *Spin*(7) spin cohomology

The $Spin(7)$ invariant spinor is $\zeta = \frac{1}{\sqrt{2}}(e_1 - e_2 \wedge e_3 \wedge e_4)$. Therefore, $\zeta \in \Delta^-$. So the spin cohomology operator is $d : \mathcal{C}_+ \to \mathcal{C}_+$.

Theorem 8. *The spin operator is nilpotent, $d^2 = 0$, if the connection ∇ is the Levi-Civita connection of a metric on the manifold M with holonomy $Spin(7)$.*

Proof. The representations $\Delta^\pm$ are real. The map $\tau : \Delta^+ \to \Lambda^1(\mathbb{R}^8)$ given by $\tau(\eta) = C\Gamma_\mu(\zeta, \eta)e^\mu = C^\mu_\zeta(\eta)e_\mu$ induces an isomorphism between the Δ^+ and the vector representations, $C = A$. This can be easily seen by observing that there is a (real) basis in Δ^+ such that τ is diagonal. This basis is

$$\begin{aligned}
&1 + e_1 \wedge \ldots \wedge e_4, \quad i(1 - e_1 \wedge \ldots \wedge e_4), \quad i(e_1 \wedge e_2 + e_3 \wedge e_4), \quad (e_1 \wedge e_2 - e_3 \wedge e_4),\\
&e_1 \wedge e_3 + e_2 \wedge e_4, \quad i(e_1 \wedge e_3 - e_2 \wedge e_4), \quad i(e_2 \wedge e_3 + e_1 \wedge e_4), \quad (e_2 \wedge e_3 - e_1 \wedge e_4).
\end{aligned} \tag{10.1}$$

In addition we have that

$$d^2\phi = \frac{1}{2} C^\mu_\zeta \bar{\wedge} C^\nu_\zeta R_{\mu\nu}\phi. \tag{10.2}$$

The right-hand-side will vanish if the curvature R of the connection ∇ is that of a Levi-Civita connection for a metric with holonomy $Spin(7)$ by virtue of the Bianchi identity. □

Corollary 9. *Let M be a manifold equipped with a metric with holonomy $Spin(7)$. Then $H^*(\mathcal{C}_+) = H^*_{dR}(M)$.*

Proof. The map τ is an isomorphism between the spin cohomology complex $(\mathcal{C}_-, d)$ and the de Rham complex $(\Lambda^*(M), d)$. Therefore, it induces an isomorphism in cohomology. □

11. G_2 spin cohomology

The spinor $Spin(7)$ representation Δ decomposes under G_2 as $\Delta = \mathbb{R} \oplus \Lambda^1(\mathbb{R}^7)$. The G_2 invariant spinor is $\zeta = \frac{1}{\sqrt{2}}(e_1 - e_2 \wedge e_3 \wedge e_4)$. As in the previous cases, one can define a linear operator d on $\mathcal{C}$ using the spinor ζ.

Theorem 9. *Let M a manifold equipped with a metric g with holonomy contained in G_2. The operator d associated to the Levi-Civita connection is nilpotent.*

Proof. Since $\Delta = \mathbb{R} \oplus \Lambda^1(\mathbb{R}^7)$, we have that $\mathcal{C}^\ell = \Lambda^\ell(M) \oplus \Lambda^{\ell-1}(M)$. Moreover, $\tau : \Delta \to \Lambda^1(\mathbb{R}^7)$ such that $\tau(\eta) = C\Gamma^\mu(\zeta, \eta)e_\mu$ is onto and has kernel $\mathbb{R}\langle\zeta\rangle$. Next

$$d^2\phi = \frac{1}{2} C\Gamma^\mu \bar{\wedge} C\Gamma^\nu \bar{\wedge} R_{\mu\nu}\phi \tag{11.1}$$

which vanishes because of the Bianchi identity where R is the curvature of the G_2 metric. □

Corollary 10. *Let M a manifold equipped with a metric with holonomy G_2. Then $H^\ell(\mathcal{C}) = H^\ell_{dR}(M) \oplus H^{\ell-1}_{dR}(M)$.*

Proof. The map τ induces an isomorphism between the two complexes. This establishes the isomorphism between the cohomologies. □

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